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Time-Varying Mixing Weights  
in Mixture Autoregressive  
Conditional Duration Models

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Revised - June 2006

## Abstract

Financial<sup>1</sup> market price formation and exchange activity can be investigated by means of ultra-high frequency data. In this paper we investigate an extension of the Autoregressive Conditional Duration (ACD) model of Engle and Russell (1998) by adopting a mixture of distribution approach with time varying weights. Empirical estimation of the Mixture ACD model shows that the limitations of the standard base model and its inadequacy of modelling the behavior in the tail of the distribution are suitably solved by our model. When the weights are made dependent on some market activity data, the model lends itself to some structural interpretation related to price formation and information diffusion in the market.

**Keywords:** Autoregressive Conditional Durations, Financial durations, Mixture of distributions, Time-varying weights.

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<sup>1</sup>Thanks are due to Estela Bee Dagum, Silvano Bordignon and Tommaso Proietti for their support and encouragement throughout this research project. Various participants in the Conference Statistical Inference on Linear and Nonlinear Dynamics in Time Series in Bressanone, June 9-11, 2005, and in the International Conference on Finance in Copenhagen, September 2-4, 2005 provided useful comments. Without implicating, we mention in particular Luc Bauwens, Nikolaus Hautsch, Søren Johansen and Timo Teräsvirta who helped us focus more on the statistical issues and economic interpretation of what is being presented here. We would also like to thank an anonymous referee who pointed out some important aspects to be clarified. Financial support from the Italian MIUR (under PRIN and FISR grants) is gratefully acknowledged.

# 1 Introduction

The dynamics of exchanges and the mechanics of price formation, in the presence of institutional arrangements and asymmetric information has received a lot of attention in the financial literature. In theoretical models, heterogeneity plays an important role in determining the size of price movements, the amount being exchanged, the frequency at which orders are presented and executed, and so on. Propositions can be tested by making use of the increasing availability of financial data at ultra-high frequency.

Trades occur at different times measured along a calendar clock, but they are accompanied by many pieces of information about the trade itself, for example, at what price it occurred and what volume was involved in the trade. Tick-by-tick data, irregularly spaced in themselves, can be translated into time series which can be fed to appropriately defined models and econometric techniques to estimate their parameters. Time duration elapsed between events observable when trades occur is a positive valued process which shows a striking persistence (clustering) similarity with other positive valued financial series such as absolute returns and volumes exchanged. This clustering is the result of market activity intensifying in certain periods of the day and thinning out in others with the result that short durations tend to be followed by short durations. The seminal paper by Engle and Russell (1998) introduced a model, named Autoregressive Conditional Duration (ACD), where durations are the result of the product between a positive valued innovation process (exponential, in the original paper) and a conditional term which has the same autoregressive behavior as in the GARCH-type models

for conditional variances.

In the basic formulation, these models consider that all heterogeneity is captured by the conditional expectation term which is linear in lagged duration and exhibits persistence decaying at an exponential rate. Results in Engle and Gallo (2006) show that in the ACD, as in any other Multiplicative Error Model, the assumption of a (constrained) Gamma distribution for the innovations entails that the first order conditions for estimation are independent of the single parameter of the Gamma. Hence the estimated ACD models assuming an exponential distribution are actually more general. The point we are making is that the suggested distributions do not manage to capture the heavy tailedness of the distribution of the innovations and we engage in a mixture assumption which does not exhibit the same shortcomings. Therefore, whether with an exponential assumption or with other distributions inserted for the innovation process, the basic formulation fails to capture the behavior of the process in the tails. This and other undesirable empirical features of the model have prompted a number of studies which have suggested a number of modifications: Ghysels et al. (2004) suggest a two factor model for financial durations and stock market activity since the two processes are interdependent; Bauwens and Giot (2000) suggest a logarithmic specification of the ACD. Some authors specifically concentrate on the modification of the conditional expectation expression of the standard ACD: e.g. Jasiak (1998) inserts fractional integration behavior and Zhang et al. (2001) make the parameters of the conditional expectation dependent on a nonlinear function of economic variables in a threshold model. Others suggested a modification of the distributional assumptions: e.g. Bauwens and Veredas (2004), insert a double stochastic pro-

cess one for the observed durations and the other for a latent variable regulating the information flow; Grammig and Maurer (2000), present a model which implies non-monotonic hazard functions by making use of the Burr distribution (a Gamma mixture of Weibull distributions).

In what follows, we adopt a mixture of exponentials with time-varying mixing weights; maintaining the hypothesis that the innovation process has unit expected value, we keep the interpretation of the conditional term being the expected duration provided by the model. We suggest that the time-varying arrival rates implied by the model represent different regimes of intensity in trading. The fact that we make this arrival rates dependent on some indicator of market activity is suggestive of a more market microstructure-based interpretation of the weights as the proportion of a certain type of trader (possessing private information and having a time-varying arrival rate) being present in the market. The results of estimating such a model on ultra high frequency data from the NYSE Trades and Quotes database show that the model fits the data well.

Here is what the reader should expect. Section 2 presents the basic ACD models and suggests how to incorporate heterogeneity in the innovation process. Section 3 studies a new formulation of the ACD model whose relevant by-product is the estimate of the time-varying arrival rates in one of the two components of the mixture. In Section 4 an application with IBM data is carried out and Section 5 concludes.

## 2 ACD models and Heterogeneity

The financial market microstructure literature (e.g. O'Hara, 1995) analyzes the mechanisms at work in the price formation process and in the interaction among market agents giving rise to market activity as it develops during market opening time. Institutional frameworks and specific rules of the exchanges must also be taken into consideration especially in what concerns the activity during market pre-opening (cf. Ghysels *et al.* 2000) and market clearing procedures (dynamics of the order book). Particular emphasis is given to the different degree of information possessed by various traders since some of the activity observed in the market can be ascribed to an imitation mechanism of agents who take price movements as disclosing private information and follow in the footsteps of the moves performed by others.

The time elapsed between one market event (be it a trade, a price movement in absolute value above a certain threshold or an accumulation of volume traded above a certain level) and the next are an obvious function of these elements, although it is not a straightforward task to identify which forces are at work and determine the outcomes. This notwithstanding, durations are an important indicator of market activity (even in their relations to price volatility) have the important feature of being temporally clustered, so that short durations tend to be followed by short durations and long by long.

In their general formulation, ACD models can be written as follows:

$$\begin{aligned}
X_i &= \phi(t_i)\Psi_i\epsilon_i \\
\Psi_i &= f(x_{i-1}, \dots, x_{i-q}, \Psi_{i-1}, \dots, \Psi_{i-p}) \\
&= \omega + \sum_{h=1}^q \alpha_h x_{i-h} + \sum_{k=1}^p \beta_k \Psi_{i-k}
\end{aligned} \tag{1}$$

where  $\epsilon_i \sim \text{iid}$  with positive support and  $E(\epsilon_i) = 1$ . They are a special case of the more general class of Multiplicative Error Models (MEM - cf. Engle and Gallo, 2005)

The subscript  $i$  refers to the  $i$ -th market event recorded at time  $t_i$ ,  $X_i$  is the  $i$ -th duration, that is  $X_i = t_i - t_{i-1}$  and  $\phi(t_i)$  is a daily seasonal component. Letting  $x_i = X_i/\phi(t_i)$  such that  $x_i$  is the seasonally adjusted duration and denoting the information set up to time  $t_{i-1}$  as  $\mathcal{I}_{i-1}$ , in the ACD model

$$E(x_i|\mathcal{I}_{i-1}) = \Psi_i,$$

so that  $\Psi_i$  is the expected duration conditionally on the information up to the time  $t_{i-1}$  and is (conditionally) deterministic.

The simplest distributional assumptions for the conditional excess durations  $\epsilon_i$  are the Exponential and the Weibull. Unfortunately, they are far from capturing the most salient features of the  $\epsilon_i$ 's, particularly their variability. Alternative hypotheses have been studied (eg. Generalized Gamma – Lunde, 1999; or Burr – Grammig and Maurer, 2000), but without managing to change considerably the tail behavior of the estimated model.

Following De Luca and Zuccolotto (2003) and De Luca and Gallo (2004) we can suggest the use of a mixture of two exponential distributions, deferring to a



later section the discussion of possible economic interpretations of this assumption. For the time being, let us assume that the excess durations, that is actual over expected durations, follow a different probability law, labelled as “I” and “U”:

$$f\left(\frac{x_i}{\Psi_i}\right) = p^I g\left(\frac{x_i}{\Psi_i}; \boldsymbol{\theta}^I\right) + (1 - p^I)g\left(\frac{x_i}{\Psi_i}; \boldsymbol{\theta}^U\right),$$

where  $\boldsymbol{\theta}^I$  and  $\boldsymbol{\theta}^U$  are the parameter vectors characterizing the densities  $g(\cdot)$  and  $0 < p^I < 1$ . The simplest assumption is a mixture of two exponential distributions for the innovation term with parameters  $\lambda^I$  and  $\lambda^U$ ,

$$f(\epsilon_i) = p^I \frac{1}{\lambda^I} \exp\left\{-\frac{\epsilon_i}{\lambda^I}\right\} + (1 - p^I) \frac{1}{\lambda^U} \exp\left\{-\frac{\epsilon_i}{\lambda^U}\right\}.$$

Imposing a unit expected value of the mixture,

$$E(\epsilon_i) = p^I \lambda^I + (1 - p^I) \lambda^U = 1.$$

involves a link between the arrival rates of the two components

$$\lambda^I = \frac{1}{p^I} [1 - (1 - p^I) \lambda^U].$$

We will denote this model as Mixture ACD (M-ACD).

### 3 The Influence of Market Conditions

De Luca and Gallo (2004) show that the Mixture Exponential ACD model performs well in capturing the tail behavior of the conditional distribution of excess durations. In the empirical section of the paper we will give additional evidence of such a feature of this model. A limitation of the model, though, is that the

assumption of constant weights in the mixture may mask the possibility of variable market conditions, characterized by varying intensity of trading. For this reason, we advance the suggestion that mixing weights may be time varying: the corresponding model will be denoted as Time-Varying Mixture (TVM) ACD. A rationale for time varying weights may be connected with the non constant arrival rate of news on the markets and/or with a varying proportion of informed traders relative to uninformed ones (cf. Ghysels, 2000), or yet with volatility clustering. More on these aspects will follow the presentation of the results: for the time being we would like to concentrate on various statistical specifications of the model, on their properties and on their performance on a financial duration time series.

Let us start by extending the notation and defining  $p_i^I + p_i^U = 1$  the weights at time  $t_i$ , both time varying. We can insert a degree of asymmetry between the two components of the mixture, distributions  $I$  and  $U$  by assuming that the inverse of the arrival rate for distribution  $I$ ,  $\lambda_i^I$ , is time dependent, while we will keep the inverse of the arrival rate of distribution  $U$ , namely  $\lambda^U$ , constant (cf., again, Ghysels, 2000). Such an assumption is not restrictive and could help us in providing a market structure interpretation to the nature of the weights.

Accordingly, the density of a mixture of two exponentials becomes

$$f(\epsilon_i) = p_i^I \frac{1}{\lambda_i^I} \exp\left\{-\frac{\epsilon_i}{\lambda_i^I}\right\} + (1 - p_i^I) \frac{1}{\lambda^U} \exp\left\{-\frac{\epsilon_i}{\lambda^U}\right\}$$

To close the model, we need an expression to describe the time evolution of  $p_i^I$ : a simple formulation for  $p_i^I$  comes from a logit framework

$$p_i^I = \frac{\exp\{\delta_0 + \boldsymbol{\delta}' \mathbf{z}_{i-1}\}}{1 + \exp\{\delta_0 + \boldsymbol{\delta}' \mathbf{z}_{i-1}\}},$$

where we make the weight dependent on a vector  $z_{i-1}$  of variables known at time  $t_{i-1}$ , there including past estimated values of  $p_i^I$  as well. A suitable choice of these predetermined variables will offer the possibility of linking time varying behavior of the weights to some observable phenomenon.

For the time being, let us concentrate on the properties of the model. If we impose a unit mean constraint on the mixture of distributions, we get

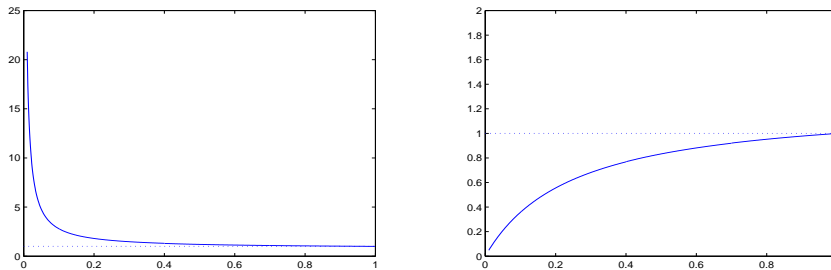
$$\lambda_i^I = \frac{1}{p_i^I} \left[ 1 - (1 - p_i^I)\lambda^U \right] = \lambda^U + \frac{1 - \lambda^U}{p_i^I},$$

known as a function of  $I_{i-1}$ , that is, as a function of  $p_i^I$  and of the constant inverse of the arrival rate in distribution  $U$ . We can establish a relationship between  $p_i^I$  and the rate of arrival in the distribution labelled  $I$ ,  $\frac{1}{\lambda_i^I}$ , depending on whether  $\lambda^U$  is smaller or greater than one. In the former case, a direct relationship holds, that is

$$\lim_{p_i^I \rightarrow 0} \frac{1}{\lambda_i^I} = 0 \text{ and, conversely, } \lim_{p_i^I \rightarrow 1} \frac{1}{\lambda_i^I} = 1,$$

or, as  $p_i^I$  increases, so does the corresponding arrival rate. The relationships of  $\lambda_i^I$  and  $\frac{1}{\lambda_i^I}$  against  $p_i^I$  are depicted graphically in Figure 1.

Figure 1: Relationship between  $\lambda_i^I$ (left) and  $\frac{1}{\lambda_i^I}$  (right) against  $p_i^I$  ( $\lambda^U < 1$ ).



When  $\lambda^U$  is greater than one, the rate of arrival in distribution  $I$  is inversely related to the weight attributed to that distribution in the mixture. However, since  $\lambda_i^I$  has to be positive, it must be that

$$p_i^I > \frac{\lambda_U - 1}{\lambda_U}.$$

In this case, the limits are

$$\lim_{p_i^I \rightarrow \frac{\lambda_U - 1}{\lambda_U}} \frac{1}{\lambda_i^I} = \infty$$

and

$$\lim_{p_i^I \rightarrow 1} \frac{1}{\lambda_i^I} = 1.$$

Figure 2 contains the plots of  $\lambda_i^I$  and  $\frac{1}{\lambda_i^I}$  against  $p_i^I$ .

The empirical evidence produced in this paper suggests however that, at least for the stock and sample period examined, the relevant case is the former, given that the distribution labelled as  $U$  has a higher arrival rate.

The assumptions described so far imply a consequence of conditional heteroskedasticity for  $\epsilon_i$ , in that

$$\text{Var}(\epsilon_i | I_{i-1}) = p_i^I (2\lambda_i^I (\lambda_i^I - 1) + 1) + (1 - p_i^I) (2\lambda^U (\lambda^U - 1) + 1). \quad (2)$$

Under an assumed  $\lambda^U$  smaller than one, we can characterize the relationship between  $p^I$  and  $\text{Var}(\epsilon_i)$  as an inverse one (cf. Figure 3 for a numerical example for  $\lambda_U = 0.5$ ). As the weight associated with distribution  $I$  decreases, the noise in excess durations increases, and there is more uncertainty around the expected durations. This mechanism allows for more flexibility in the tail behavior (in our example, as we will see, for estimated  $\lambda_U$  approximately equal to 0.5, the

empirical estimates of  $p_i^I$  oscillate around 0.25 which in turn forces the variance to oscillate around 2.5): the data will attribute a higher weight to the distribution  $I$  to capture a lower variance in the expected durations, while a higher weight to distribution  $U$  will be associated with a higher variance.

Figure 2: Relationship between  $\lambda_i^I$  (left) and  $\frac{1}{\lambda_i^I}$  (right) against  $p_i^I$  ( $\lambda^U > 1$ ).

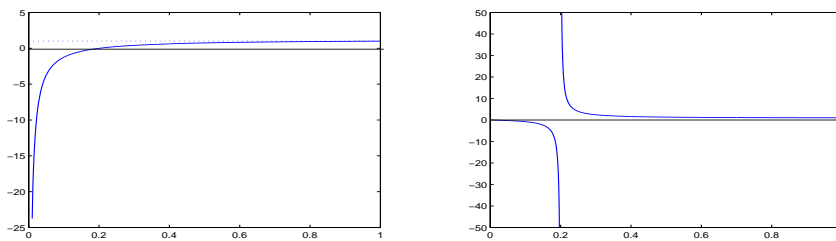
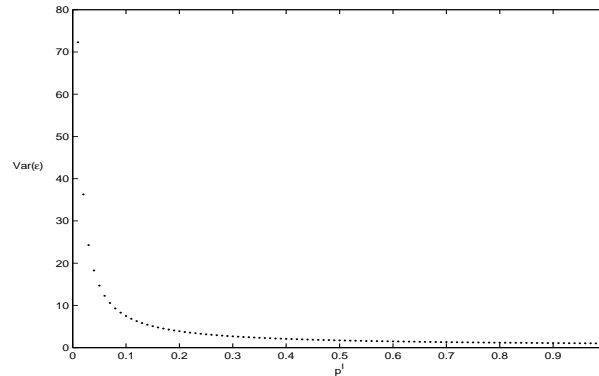


Figure 3: Relationship between  $p_i^I$  and  $\text{Var}(\epsilon_i)$ .



In order to make sure that the effects captured by our mixture ACD model are truly affecting the variance and are not due to a misspecification of the mean,

we will insert the market related variables in the expression for the conditional expected duration  $\Psi_i$ .

Summarizing, the model to be estimated departs in various ways from the standard ACD (cf. (1) above)

$$\begin{aligned}
x_i &= \Psi_i \epsilon_i \\
\Psi_i &= \omega + \sum_{h=1}^q \alpha_h x_{i-h} + \sum_{k=1}^p \beta_k \Psi_{i-k} + \gamma' z_{i-1} \\
\epsilon_i &\sim \text{ME}(\lambda_i^I, \lambda^U, p_i^I)
\end{aligned} \tag{3}$$

The assumption that the mixing innovation  $\epsilon_i$  has a unit mean implies also in this model that  $E(x_i|I_{i-1}) = \Psi_i$ , so that  $\Psi_i$  keeps its interpretation of conditional expected duration possibly with additional effects coming from the predetermined variables. As per the variance, we have  $\text{Var}(x_i|I_{i-1}) = \Psi_i^2 \text{Var}(\epsilon_i)$ . As seen,  $\text{Var}(\epsilon_i)$  depends on  $p_i^I$  which is a function of the predetermined variables  $z_{i-1}$  in the logit specification, to be suitably chosen, and the different arrival rates in the two exponential distributions.

In the framework that we pursue here, that is durations between price movements beyond a certain threshold, we can resort to market conditions observable within the time duration, recognizing that in between the price movements relevant for determining the durations there may occur other transactions which are not associated to substantial price increases or decreases. We can build two such variables:

1. the trading intensity at time  $t_i$ ,  $TI_i$ , is defined as the ratio of the number of trades recorded during the duration  $x_i$  and the duration itself. A positive coefficient in the conditional expectation expression would mean that the

variable has an effect to increase the expected time elapsed between subsequent relevant price movements. A positive coefficient in the logit specification would signal that a higher number of such trades per unit of time increases the weight attributed to distribution  $I$  and decreases the variance around the expected duration. The opposite would be true if the coefficient were negative.

2. For a duration  $x_i$ , the average volume,  $AV_i$ , is defined as the ratio of the traded volume over the number of transactions. Also in this case, a positive coefficient would associate a higher average volume with longer expected durations in the conditional mean, whereas it would have the effect to decrease the variance if the coefficient in the logit specification were positive and the effect to increase the variance if the coefficient is negative.

Timing is such that these variables need to enter with a lag in the specification.

## 4 IBM data

The empirical analysis is focused on the IBM stock with price data coming from the Trades and Quotes database of the NYSE. The duration data is built for transactions involving a (cumulative) price change greater than  $\frac{1}{16}$  of one dollar. The observations chosen start on March 02, 2000 and end on March 17, 2000 for a total of 14 trading days. We selected the transactions between 10:00AM and 4:00PM. We opted for a short period in order to illustrate the functioning of our model avoiding the possibility of nonstationarities in the behavior of the series

on a longer horizon. The total number of transactions (12707) was obtained after carrying out some cleaning operations following the procedure described by Brownlees and Gallo (2005). Durations involving the first price change of a day and the last price change of the previous day were also deleted.

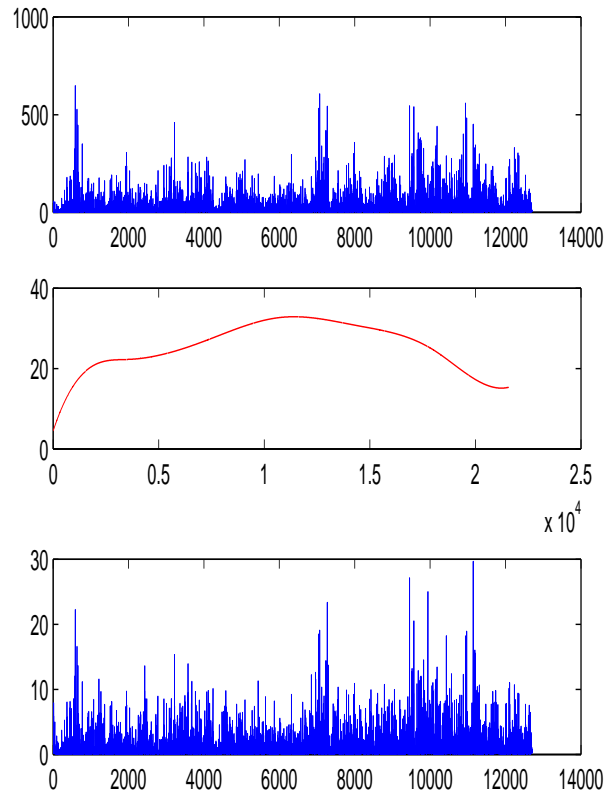
Let us start by providing some descriptive evidence in the behavior of the series which will be used in the estimations. We start by showing in Figure 4 the raw durations, the estimated daily seasonal component (estimated through a cubic spline with nodes set at each hour) and the adjusted durations. The clustering of adjusted durations is shown in Figure 5 where we report the estimated total and partial autocorrelation functions. The autocorrelation function depends on the parameters of the expected duration (Bauwens and Giot, 2001, p.71). Some evidence of long-range dependence in the durations has been advanced by Jasiak (1998) who has inserted a fractionally integrated term in the parameters. Sun et al. (2005) have adopted some stable distribution. It seemed to us that a comparison of the performance of our model with the class of models which adopt alpha-stable innovations is beyond the scope of the current paper.

As a preliminary justification for what follows, let us start from the estimation of a standard ACD model with an exponential assumption for the innovation term. Preliminary testing shows that an ACD(1,2) is advisable and this choice of orders for the model will be kept throughout. As shown in Table 1, the customary excess variance in the estimated residuals is obtained also in our case.

Let us move now to the consideration of the model with time varying weights, and let us start by showing some evidence about the behavior of the two variables chosen to represent market conditions as expectations on future durations are to



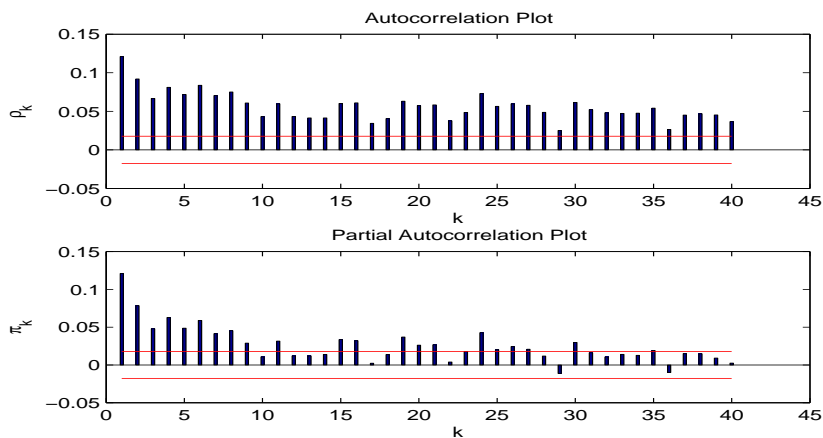
Figure 4: The durations data (top), the estimated daily seasonal component (middle) as a function of the time of day (expressed in seconds) and the adjusted durations (bottom).



be formed.

In Figure 6 we report the time series graph of the Trading Intensity ( $TI_i$ ) defined as the ratio of the number of trades recorded between  $t_{i-1}$  and  $t_i$  over

Figure 5: Estimated total and partial autocorrelation functions for adjusted durations.



the length of the seasonally adjusted duration,

$$TI_i = \frac{NT_i}{x_i}.$$

In Figure 7 we report the time series graph of the Average Volume ( $AV_i$ ) built as the ratio of volume recorded between  $t_{i-1}$  and  $t_i$  (divided by 10000) over number of trades recorder during the same period

$$AV_i = \frac{Vol/10000}{NT_i}.$$

Table 2 reports some summary statistics about adjusted durations, number of trades within duration, trading intensity and average volume. The highest correlation is between adjusted durations and number of trades, with the other variables exhibiting very low linear correlation with one another.

In order to keep the comparison manageable and interpretable, we will consider four models where the mixing distributions are inserted, identified by the

Table 1: Estimation of an Exponential ACD.

| Parameters estimation ( <i>standard errors</i> ) |                 |
|--|-----------------|
| Parameter  | ACD Model       |
| $\omega$   | 0.0223 (0.0033) |
| $\alpha$   | 0.0624 (0.0051) |
| $\beta_1$  | 0.4386 (0.0782) |
| $\beta_2$  | 0.4775 (0.0752) |
| Statistics on residuals                          |                 |
| Statistics                                       | ACD Model       |
| Mean   | 1.000           |
| Variance   | 2.558           |
| Theoretical Var                                  | 1               |
| $Q(10)$  | 14.42           |

presence or the absence of a number of features. We may have:

- Fixed versus Time-varying weights;
- Presence of predetermined variables in the  $\Psi_i$  specification (*Extended  $\Psi_i$* ), and/or in the  $p_i^I$  specification: Trading Intensity ( $TI_i$ ) and Average Volume ( $AV_i$ ) as defined above.

We will clearly mark in each of the Tables that follow what features are present in each model. We have gathered further evidence about what is involved when the assumption of unit expectation of the innovation  $\epsilon_i$  is removed (i.e. when the

Figure 6: Trading intensity  $TI_i$ .

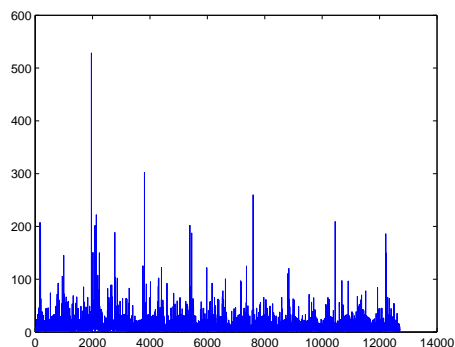
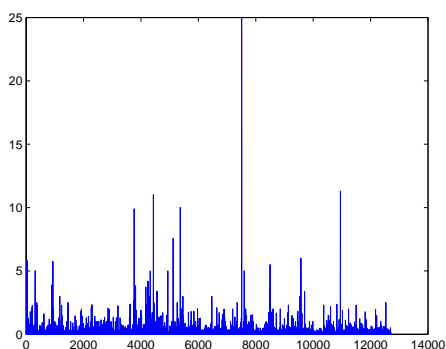


Figure 7: Average volume per trade,  $AV_i$ .



arrival rate of distribution  $I$  becomes constant). The results are not reported for the sake of space but are available on request.

Table 3 compares two specifications: Model 1 is the De Luca and Gallo (2004) specification where the adoption of a mixture is made with fixed weights. As a preliminary check<sup>2</sup> we estimate a corresponding model (Model 2) with fixed mixing weights extending the number of variables included in the specification

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<sup>2</sup>We thank Timo Teräsvirta for pointing this out to us.

Table 2: Descriptive statistics on variables of interest

| Statistics  | Adj. Dur. | NT    | TI      | AV     |
|-------------|-----------|-------|---------|--------|
| Min         | 0.030     | 1     | 0.298   | 0.010  |
| Mean        | 1.027     | 5.802 | 9.953   | 0.143  |
| Max         | 29.669    | 283   | 528.306 | 25     |
| St. dev.    | 1.736     | 9.708 | 13.285  | 0.404  |
| Correlation |           |       |         |        |
| Adj. dur.   | -         | 0.832 | -0.192  | -0.016 |
| NT          |           | -     | -0.005  | -0.034 |
| TI          |           |       | -       | -0.044 |

for  $\Psi_i$ . The two models are defined as follows:

$$x_i = \Psi_i \epsilon_i$$

$$E(\epsilon_i = 1), \quad \epsilon_i \text{ mixture of exponentials with fixed weights}$$

$$\text{Model 1 : } \Psi_i = \omega + \alpha x_{i-1} + \beta_1 \Psi_{i-1} + \beta_2 \Psi_{i-2}$$

$$\text{Model 2 : } \Psi_i = \omega + \alpha x_{i-1} + \beta_1 \Psi_{i-1} + \beta_2 \Psi_{i-2} + \gamma_1 TI_{i-1} + \gamma_2 AV_{i-1}$$

The diagnostics on the second model will point out whether some substantial modification in the variance of the standardized residuals is detected.

The likelihood-ratio test shows a significance of the presence of the market related variables in the  $\Psi_i$  specification with negative coefficients pointing to a reduction of the expected duration. The diagnostics on the residuals is not altered, except for a reduction in the serial autocorrelation signalled by the Ljung-Box Q

statistic at various lags.

In estimating time-varying weights we report only the results for a logit model for the weights specified as

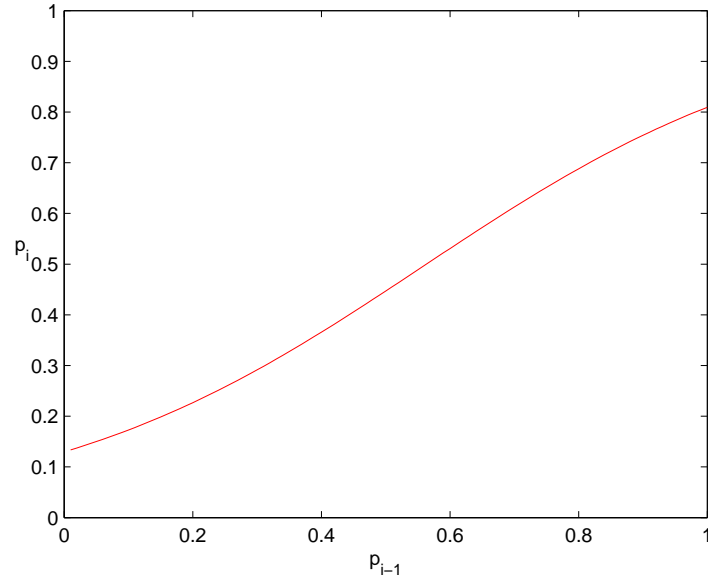
$$p_i^I = \frac{\exp\{\delta_0 + \delta_1 TI_{i-1} + \delta_2 p_{i-1}^I\}}{1 + \exp\{\delta_0 + \delta_1 TI_{i-1} + \delta_2 p_{i-1}^I\}},$$

Other specifications without the autoregressive term performed substantially less satisfactorily, and the variable  $AV_{i-1}$  systematically lacked statistical significance.

Consistently, the insertion of the market related variables in the specification with time-varying mixing weights proves to be statistically significant (cf. Table 4). The residual diagnostics is quite satisfactory, in that the variance of the residuals standardized by subtracting the theoretical unit mean and dividing by the time-varying standard deviation is close to one and the residual autocorrelation is limited. The chosen specification for the time varying weights is quite persistent: the implied estimated relationship between  $p_{i-1}^I$  and  $p_i^I$  when the variable  $TI_{i-1}$  is set at the sample mean is shown in Figure 8.

Figure 9 shows the estimated behavior of time varying mixing weights  $p_i^I$  for the Model 4, in which trading intensity and average volume enter the specification of the conditional expectation and trading intensity and an autoregressive term regulate the behavior of the logit mapping. Vertical bars separate the days of the sample. A daily seasonal pattern is made more apparent by plotting the estimated values for the various days separately, as in Figure 10: in particular, the highest values of the  $\hat{p}_i^I$  occur in correspondence with the beginning of the trading day. A drop is observed in the mid-day time. Sometimes a rise is detected in the last hours. Fitting some polynomials of time to the daily results shows a bowl shaped

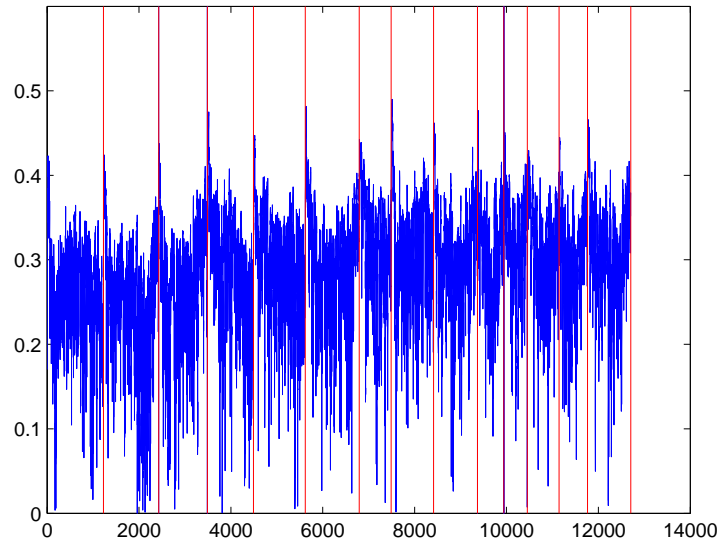
Figure 8: Relationship between  $p_{i-1}^I$  and  $p_i^I$ : Estimated Logit relationship with TI fixed at the sample mean



pattern for most of the days.

Correspondingly, we can show the time-varying behavior of the estimated variance of the residuals coming from the estimated ACD model with mixing weights. In Figure 11 we have to exclude the highest spikes and adopted a logarithmic scale, in order to appreciate the oscillations of the series. Note that in this case the oscillations vary in excess of 2, while in case of the fixed weights mixture the variance was estimated at around 2.53 and should be interpreted as an unconditional estimate of what we have estimated with time-varying weights. We take

Figure 9: Estimated mixing weight for distribution  $I$ . Model with time varying weights, unit expected innovation and extended  $\Psi_i$  specification.



this to be further evidence of the flexibility provided by our model in capturing time-varying nature of market dynamics.

## 5 Concluding Remarks

In this paper we extend previous findings (De Luca and Gallo, 2004) on the suitability of a mixture of distribution assumption for the modelling of durations between relevant price movements within an ACD framework. We have shown the statistical properties of a model in which two exponential distributions are used with time-varying weights, keeping the arrival rate in one of the distributions to



be constant and the other one to be time-varying. The evolution of the weights is assumed autoregressive in a logit framework and is taken to be dependent on some observable variables linked to some features of market activity in between relevant price movements. The results are quite encouraging based on the statistical significance of the estimated coefficients and on the features of the estimated time-varying variance of the innovation term.

We did not engage in a micro-structural motivation of our model or an explicit interpretation of its findings in terms of price formation dynamics. There are a few features that are worth mentioning at this stage which deserve further attention with a richer dataset which would allow us to mark each trade as a “buy” or a “sell”. The idea of a mixture of two distributions conjures up images of two types of traders or two types of regimes. We have purposely marked our two distributions as  $I$  and  $U$  because a promising structural explanation for our model would be the presence in the market of traders who have asymmetric information (good or bad news – informed) and traders who trade for liquidity (uninformed). The literature in this respect is very rich and can be traced to the work of Easley, O’Hara and their co-authors (e.g. Easley and O’Hara, 1987; Easley et al., 1996; Easley, et al., 2003). Some of our assumptions and findings are suggestive in that direction: for example, the time-varying arrival rate in the distribution of type  $I$  is consistent with the presence of traders who move on the basis of news events available at irregular intervals within the day; the seasonal intra-daily pattern which we find in the weight attributed to type  $I$  is consistent with several findings (e.g. Nyholm, 2002) that the information content of the trading process is larger in the morning than in the afternoon; the fact that our model predicts that variance in the inno-

vation would increase if more weight is to be given to the type  $U$  distribution is consistent with the fact that the presence of uninformed (i.e noisy) traders makes prediction on future expected durations less precise.

From a more statistical point of view we feel that the extension suggested in this paper is a very important one, since it introduces the possibility of modelling mixing parameters in a time-varying fashion. What we have accomplished is to estimate these parameters and investigate whether the inclusion of explanatory variables in the determination of the weights sharpens the picture. The results we obtain on the IBM dataset are quite encouraging, since we improve upon the standard ACD model and we show the gains that one obtains over the fixed weights case.

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Table 3: Estimation of ACD Models with fixed mixing weights

| Parameter                           | Model 1                              | Model 2          |
|-------------------------------------|--------------------------------------|------------------|
| TV weights                          | no                                   | no               |
| $E(\epsilon_i = 1)$                 | yes                                  | yes              |
| Extended $\Psi_i$                   | no                                   | yes              |
| Specification for $\Psi_i$          |                                      |                  |
| $\omega$                            | 0.0315 (0.0058)                      | 0.0742 (0.0078)  |
| $\alpha$                            | 0.0619 (0.0070)                      | 0.0568 (0.0066)  |
| $\beta_1$                           | 0.3518 (0.1007)                      | 0.4751 (0.1146)  |
| $\beta_2$                           | 0.5551 (0.0971)                      | 0.4127 (0.1110)  |
| $\gamma_1$                          |                                      | -0.0015 (0.0001) |
| $\gamma_2$                          |                                      | -0.0260 (0.0107) |
| Specification for mixture           |                                      |                  |
| $p^I$                               | 0.3015 (0.0120)                      | 0.2982 (0.0120)  |
| $\lambda^U$                         | 0.4534 (0.0113)                      | 0.4580 (0.0113)  |
| $\lambda^I$                         | Function of $\lambda_U$ and of $p_i$ |                  |
| $n$                                 | 12707                                | 12707            |
| Mean log-lik                        | -0.865593                            | -0.862167        |
| Mean                                | 1.000                                | 1.000            |
| Variance                            | 2.558                                | 2.553            |
| $Q(1)$                              | 4.646                                | 3.894            |
| $Q(10)$                             | 18.22                                | 11.39            |
| $Q(20)$                             | 30.54                                | 21.67            |
| LR test (Model 2 vs Model 1): 87.07 |                                      |                  |

Table 4: Estimation of ACD Models with time-varying mixing weights

| Parameter                  | Model 3          | Model 4          |
|----------------------------|------------------|------------------|
| TV weights                 | yes              | yes              |
| $E(\epsilon_i = 1)$        | yes              | yes              |
| Extended $\Psi_i$          | no               | yes              |
| Specification for $\Psi_i$ |                  |                  |
| $\omega$                   | 0.0360 (0.0067)  | 0.0614 (0.0087)  |
| $\alpha$                   | 0.0555 (0.0068)  | 0.0488 (0.0068)  |
| $\beta_1$                  | 0.3549 (0.1119)  | 0.4835 (0.1412)  |
| $\beta_2$                  | 0.5559 (0.1078)  | 0.4219 (0.1360)  |
| $\gamma_1$                 | -                | -0.0011 (0.0002) |
| $\gamma_2$                 | -                | -0.0234 (0.0100) |
| Specification for mixture  |                  |                  |
| $\delta_0$                 | -1.6476 (0.1783) | -1.7329 (0.2380) |
| $\delta_1$                 | -0.0256 (0.0059) | -0.0216 (0.0072) |
| $\delta_2$                 | 3.3971 (0.4228)  | 3.5383 (0.5887)  |
| $\lambda^U$                | 0.4385 (0.0108)  | 0.4502 (0.0111)  |
| $\lambda^I$                | Time Varying     |                  |
| Mean log-lik               | -0.861913        | -0.860168        |
| Mean                       | -0.007           | -0.006           |
| Variance                   | 1.018            | 1.017            |
| $Q(1)$                     | 6.465            | 8.149            |
| $Q(10)$                    | 22.68            | 20.30            |
| $Q(20)$                    | 32.51            | 29.51            |

LR test, Model 4 vs Model 3: 44.34.

Figure 10: Breakup of the intra-daily estimates of time-varying weights  $\hat{p}_i^I$  estimated in Model 4 with time varying weights, unit expected innovation and extended  $\Psi_i$  specification. The continuous line is a quadratic function of time.

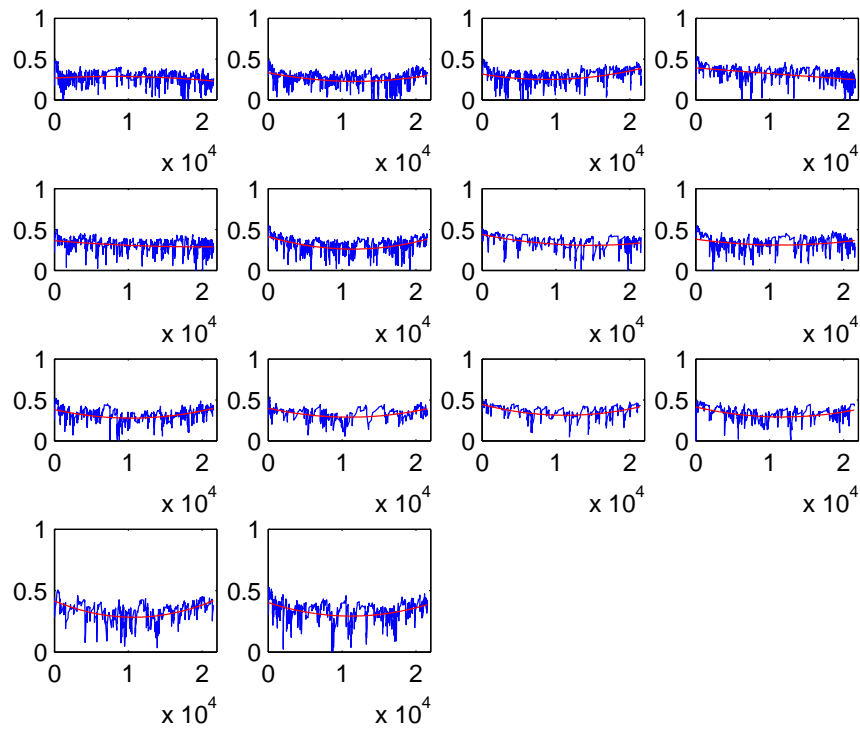
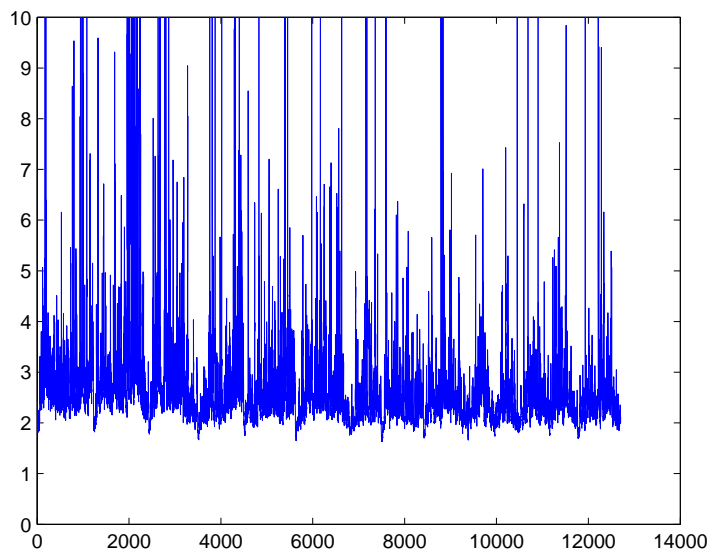


Figure 11: Estimated variance of excess durations (log-scale).





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