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Using secondary outcomes and covariates to sharpen inference in randomized experiments with noncompliance

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Using secondary outcomes and covariates to sharpen inference in randomized experiments with noncompliance[1]

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Abstract: Restrictions implied by the randomization of treatment assignment on the joint distribution of a primary outcome and an auxiliary variable are used to tighten nonparametric bounds for intention-to-treat effects on the primary outcome for some latent subpopulations, without requiring the exclusion restriction assumption of the assignment. The auxiliary variable can be a secondary outcome or a covariate, while the subpopulations are defined by the values of the potential treatment status under each value of the assignment. The derived bounds can be used to detect violations of the exclusion restriction and the magnitude of these violations in instrumental variables settings. It is shown that the reduced width of the bounds depends on the the strength of the association of the auxiliary variable with the primary outcome and the compliance status. We also show how the setup we consider offers new identifying assumptions of intention-to-treat effects without the exclusion restriction. The use of the bounds is illustrated in two real data examples of a social job training experiment and a medical randomized encouragement study.

Keywords: Nonparametric bounds, ITT effects, noncompliance, instrumental variables, multiple outcomes, covariates, violation of exclusion restrictions.

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1 Introduction

Randomized experiments often suffer from noncompliance, so that the treatment subjects actually receive is different from their randomly assigned treatment. Randomized experiments with noncompliance are closely related to the econometric instrumental variables (IV) setting, where the instrument plays the parallel role of treatment assignment. The setup we consider is one with heterogenous effects, as developed in Imbens and Angrist (1994) and Angrist et al. (1996). The approach to adjust for noncompliance applied in Angrist et al. (1996) can also be viewed as a special case of *principal stratification* (Frangakis and Rubin, 2002), where the latent subpopulations of compliers, never-takers, always-takers and defiers are the principal strata with respect to the post-assignment compliance behavior.

Given the setup, it can be shown that causal effects of assignment among compliers (subjects who would take the treatment if offered to take it and would not take it if not offered) are identifiable under the assumptions that a) treatment assignment (i.e., the instrument) is randomized, b) the assignment has a non null effect on treatment receipt (relevance of the instrument) c) there are no subjects who would take the treatment if randomized not to take it, but would not take it if assigned to take it (no-defier or monotonicity assumption), and d) under the crucial exclusion restriction assumption of a null effect of assignment on the outcome for those whose treatment status is not affected by assignment, i.e., the noncompliers, usually distinguished in never-takers and always-takers. These assumptions, which essentially define the validity of an instrument, allow one to uniquely disentangle the observed distribution of the outcome, which is a mixture of distributions associated with the latent groups of compliers, never-takers, and always-takers (Imbens and Rubin, 1997a). The effect on compliers is usually interpreted as the causal effect of receipt of the treatment, under the additional assumption that the effect of assignment for compliers is solely due to the actual treatment receipt. With a valid instrument, sharp bounds on the average treatment effect have also been derived (e.g., Manski, 1990, 1994; Balke and Pearl, 1997).

Depending on the empirical setting, some of these substantive assumptions may be questionable. Here we focus on the violation of the exclusion restriction (ER). The ER is never satisfied by design, and requires judgment of subject matter knowledge. In many cases there are reasons to doubt the ER and to assess its plausibility. Typically, ERs appear plausible in blind or double-blind placebo-controlled experiments: if subjects do not know their initial assignment, it is reasonable to argue that assignment can affect their outcome only through the effect of treatment received. It may however be questionable in open-label experiments, in randomized encouragement studies (Hirano et al., 2000), and in observational studies with instrumental variables. Open-label experiments are the norm in the social sciences, where subjects, as well as experimenters, cannot be blinded the treatment received because they actively participate to the treatment (e.g., Jo, 2002, Duflo et al., 2008), e.g., training program they are offered (e.g., Zhang et al. 2009), and in general assignment can affect the outcome through channels other than the treatment (e.g., Mattei and Mealli, 2007; Chetty et al., 2010). The exclusion restriction of an instrument on certain outcomes is also usually subject to large debates in observational studies (e.g., Angrist, 1990; Angrist and Kruger, 1991; Hoogerheide et al., 2007).

We take an approach that is closer in spirit to Hirano et al. (2000), in the sense that we focus on partial identification of intention-to-treat (ITT) effects, that is, the effects of the assignment on the subpopulations of compliers, never-takers and always-takers. Hirano et al. (2000) conduct a Bayesian analysis and assess sensitivity to various violations of the ER, by estimating *local* ITT effects in a randomized encouragement study concerning the effects of inoculation for influenza. They found that positive estimates of the overall ITT effect need not be due to the treatment itself, but rather to the encouragement to take the treatment (they found a *direct* effect of the assignment on the outcome *not through* the treatment).

When the ER is violated, and the analysis is not augmented with additional assumptions, *local* ITT effects can only be partially identified. In the literature, bounds on some of these effects have been derived (Richardson et al., 2011; Huber and Mellace, 2011), borrowing nonparametric bounds derived for so-called principal strata *direct* effects (Zhang and Rubin, 2003; Imai, 2008; Lee, 2009; Mattei and Mealli, 2011)[2]. Specifying parametric distributions for a continuous outcome

[2]A related literature on partial identification of direct effects derives bounds for alternative definition of direct effects, such as *natural* and *controlled* direct effects (Cai et al., 2008; Sjolander, 2009; Imai et al., 2010). These effects involve a priori conterfactual quantities, that we explicitly avoid using here, so that bounds derived in those papers are not relevant to us.

usually lead to point identify ITT effects, in the sense that unique MLEs exist. However, for binary outcomes, the likelihood function is flat around its maximum even in a parametric setting, i.e., assuming outcomes follow a Bernoulli distribution (Imbens and Rubin, 1997b). We focus on binary outcomes: Unlike Hirano et al. (2000), but also unlike Manski and Pepper (2000) and Flores and Flores-Lagunes (2010)[3], we do not use any additional assumption, neither in the form of prior information and distributional assumptions, nor in the form of alternative weak monotonicity assumptions, but rather use the additional information provided by the joint distribution of the outcome of interest with secondary outcomes or covariates. Tighter sharp bounds on ITT effects are derived, and specifically on the distribution of potential outcomes by compliance type and assignment values; ITT effects are in fact defined as general contrasts of features of the distribution of potential outcomes under different values of the assignment. We can show that, for binary outcomes, our bounds coincide with maximum likelihood regions.

Specifically, we exploit restrictions on the joint distribution of the primary outcome with an auxiliary variable (a covariate or a secondary outcome) implied by the randomization of treatment assignment and by the ER on the secondary outcome when this assumption is more plausible, and show that the reduced width of the bounds depends on the the strength of the association of the auxiliary variable with the primary outcome and the compliance status.

Our use of secondary outcomes and covariates is novel. In observational studies, covariates are usually used to make identifying assumptions more plausible if stated conditional on them (Manski, 1990; Abadie, 2003; Frolich, 2006; Hong and Nekipelov, 2010): under these assumptions, bounds on conditional quantities are derived and then averaged over the distribution of covari-

[3]Manski and Pepper (2000) study partial identification of the average treatment effect, when the usual ER does not hold and it is replaced by a weaker monotone instrumental variable assumption. A similar approach is followed by Flores and Flores-Lagunes (2010), who derive bounds on the local average treatment effect (LATE), i.e., the effect of the treatment for compliers, without assuming the ER, but investigating different sets of assumptions imposing weak-inequality restrictions on the mean potential outcomes. A related strand of the literature develops sensitivity analysis of IV estimates in linear models under local violations of the ER, using prior information on a parameter summarizing the extent of the violation (see Conley et al., 2008, 2011, and, similarly, Nevo and Rosen, 2011 and Kraay, 2010). A sensitivity analysis for structural slope coefficients using overidentifying restrictions is also provided by Small (2007).

ates (e.g., Lechner and Melly, 2010). Alternatively, assuming that conditional effects are constant, bounds are tightened by intersecting bounds on conditional quantities (Manski, 1990). Lee (2009) seems to be one of the few papers where the contribution of the covariates in tightening bounds is shown explicitly in a context where conditioning on covariates is not required by the assumptions.

In randomized experiments, covariates are usually conditioned on in order to improve the precision of causal estimates, by improving the prediction of the compliance status and the missing potential outcomes (Hirano et al., 2000). We show, however, that using the covariates can not only increase precision, but it can also tighten the bounds. Specifically, randomization of treatment assignment implies that the distribution of covariates is the same for the two assignment values within the subpopulations, and show how this piece of information helps sharpening the bounds of ITT effects on the outcome of primary interest.

Our result using covariates essentially exploits the independence of covariates and assignment within subpopulations. By analogy, one may argue that also secondary outcomes, for which the ER is plausible, may serve the same goal of tightening bounds. We show that the ER on a secondary outcome implies restrictions on the joint distribution of the two outcomes that can be used to sharpen bounds on ITT effects on the primary outcome. The ER is often more plausible for secondary outcomes (rather than for primary outcomes) for which the study was not specifically designed. For example, in open-label randomized experiments, the ER on secondary outcomes such as side-effects is usually plausible, because some side-effects can manifest themselves only if treatment is actually received. In some empirical settings, the contribution of a secondary outcome can be even larger than the one of a covariate, because some outcomes can be strongly associated with one other and with the compliance status, and bounds are tighter the stronger this association.

Our use of secondary outcomes differs from common practice. Usually, in the presence of multiple outcomes, analysis is conducted separately for one outcome at a time, and the joint analysis of two (or more) outcomes is not pursued, unless analyzing their association is the goal. Joint analysis of multiple outcomes is sometimes used to address issues of adjustments for multiple comparisons (e.g., Hsu, 1996).

The setup we consider is one with a binary random assignment, a binary treatment, and binary outcomes and covariates. This should not be viewed as a limit of our framework. Our results can in fact be used to point-wise bound the cumulative distribution function of a continuous outcome Y for different levels of the outcome, i.e., to derive bounds on the probabilities of the events $Y \leq y$ for each $y \in \mathcal{Y}$, where $\mathcal{Y}$ is the support of the outcome variable Y.

In what follows, we first introduce our framework and notation (Section 2). We then review and derive, in Section 3, partial identification results of ITT effects on a single outcome, with and without exclusion restriction assumptions. In Section 4 sharp tighter bounds on the ITT effects on the primary outcomes are derived. In Section 5 two limiting cases are analyzed, under which bounds collapse. Section 6 introduces another limiting case that can be used as an identifying condition for ITT effects and proposes an estimator under this assumption. Section 7 shows the identifying power of our bounds in two illustrative real data examples of a social job training experiment and a medical randomized encouragement study, where the ER for the randomized assignment may be questionable. Some concluding remarks are offered in Section 8.

2 Framework and Notation

Let introduce the potential outcome notation. Throughout the paper we will make the stability assumption that there is neither interference between units nor different versions of the treatment (SUTVA; Rubin, 1978). Under SUTVA, let Z_i be a binary treatment assignment for unit i ($Z_i = 0$ if unit i is assigned to the control group, $Z_i = 1$ if unit i is assigned to the treatment group). We denote by $D_i(z)$ the binary treatment receipt for unit i (1 = treatment, 0 = control) when assigned treatment z. $D_i(Z_i)$ denotes the actual treatment received. The two potential indicators $D_i(0)$ and $D_i(1)$ describe the compliance status and define four subpopulations: compliers (c), for whom $D_i(z) = z$ for $z = 0, 1$; never-takers (n), for whom $D_i(z) = 0$ for $z = 0, 1$; always-takers (a), for whom $D_i(z) = 1$ for $z = 0, 1$; and defiers (d), for whom $D_i(z) = 1 - z$ for $z = 0, 1$ (Angrist et al., 1996). Because only one of the two potential indicators of treatment receipt is observed, these four subpopulations are latent, in the sense that in general it is not possible to identify the specific subpopulation a unit i belongs to. We denote as G_i the subpopulation membership, which takes on values in $\{c, n, a, d\}$. We define four potential outcomes for a bivariate binary outcome,$\mathbf{Y}_i(z, d) = [Y_{i1}(z, d), Y_{i2}(z, d)]'$, for all possible combinations of treatment assignment and treatment received ($z = 0, 1$; $d = 0, 1$). However, given the compliance status, only two of the four potential outcomes are potentially observed, namely, $\mathbf{Y}_i(z, D_i(z))$, $z = 0, 1$, the other two potential outcomes being *a priori counterfactuals*. In order to avoid the use of such counterfactuals, we let the binary outcome variables depend only on treatment assignment: $\mathbf{Y}_i(z) = [Y_{i1}(z), Y_{i2}(z)]'$. In our setting, the first outcome will be considered as the outcome of primary interest and the second outcome as an auxiliary variable. We will also consider cases where the auxiliary variable is a binary covariate, X_i, thus using the joint distribution of $Y_{i1}(z)$ and X_i, $z = 0, 1$.

In what follows we will maintain the following assumptions:

Assumption 1 *Random assignment Z_i is randomly assigned, implying that*

$$Z_i \perp\!\!\!\perp D_i(1), D_i(0), \mathbf{Y}_i(1), \mathbf{Y}_i(0), X_i, \quad \forall i$$

Assumption 2 *Nonzero effect of Z on D.* $E(D_i(1) - D_i(0) \neq 0$.

Assumption 3 *Monotonicity of compliance. $D_i(1) \geq D_i(0)$, $\forall i$, which rules out the presence of defiers.*

Assumption 3 implies that the population is only composed of compliers (c), never-takers (n) and always-takers (a); we denote as π_c, π_n, and π_a the proportions of c, n and a in the target population, respectively. Assumptions 2 and 3 imply that $\pi_c \neq 0$.

We introduce the following notation for the joint distribution of potential outcomes:

$$P[\mathbf{Y}_i(z) = (y_1, y_2)|G_i = g, Z_i = z] = P[\mathbf{Y}_i(z) = y_1y_2|G_i = g, Z_i = z] = P_{gz}^{(y_1y_2)} \tag{1}$$

for $y_1y_2 = \{00, 01, 10, 11\}$, $z = \{0, 1\}$, $g = \{c, n, a\}$, and for the corresponding marginal distributions:

$$P[Y_{i1}(z) = y_1|G_i = g, Z_i = z] = P_{gz}^{(y_1\cdot)}, \tag{2}$$

$$P[Y_{i2}(z) = y_2|G_i = g, Z_i = z] = P_{gz}^{(\cdot y_2)}. \tag{3}$$

For the secondary outcome we will maintain the following stochastic exclusion restriction assumption for always-takers and never-takers:

Assumption 4 *Partial stochastic exclusion restriction.* $P^{(\cdot 1)}_{n1} = P^{(\cdot 1)}_{n0}$ *and* $P^{(\cdot 1)}_{a1} = P^{(\cdot 1)}_{a0}$.

For a covariate, note that, due to random assignment (Assumption 1), $Z_i \perp\!\!\!\perp X_i | D_i(1), D_i(0)$, $\forall i$. This implies that $P[X_i = 1|Z_i = 0, G_i = g] = P[X_i = 1|Z_i = 1, G_i = g]$ $\forall g$, $\forall i$, and this equality can be interpreted as a form of stochastic exclusion restriction which holds by design, i.e., by the randomization of the instrument, for covariates within all three latent subpopulations.

We focus on identifying intention-to-treat (ITT) effects on the first outcome, Y_1, for the subgroups of compliers, never-takers and always-takers, which are defined as:

$$E[Y_{i1}(1) - Y_{i1}(0)|G_i = g] = P^{(1\cdot)}_{g1} - P^{(1\cdot)}_{g0} \quad g = c, n, a. \tag{4}$$

ITT effects for always-takers and never-takers reflect the effect of the assignment/instrument only and can thus highlight possible violations of the exclusion restriction on the primary outcome. Differently, the ITT effect for compliers includes both the effect of assignment and the effect of treatment, and so provides information on their joint magnitude.

The data we can observe are Z_i, X_i, $D^{obs}_i = D_i(Z_i)$ and $\mathbf{Y}^{obs}_i = \mathbf{Y}_i(Z_i)$, so that the distributions that are asymptotically revealed by the sampling process are the following:

$$P[\mathbf{Y}^{obs}_i = y_1 y_2 | Z_i = z, D^{obs}_i = d],$$

$$P[Y^{obs}_{i1} = y_1 | Z_i = z, D^{obs}_i = d],$$

$$P[Y^{obs}_{i2} = y_2 | Z_i = z, D^{obs}_i = d],$$

$$P[X_i = x | Z_i = z, D^{obs}_i = d],$$

$$P[D^{obs}_i = d | Z_i = z]$$

for $y_1 = \{0, 1\}$, $y_2 = \{0, 1\}$, $z = \{0, 1\}$, $d = \{0, 1\}$, $x = \{0, 1\}$. We assume these distributions are known or can be consistently estimated, thereby not taking account of specific statistical problems related to inference in finite samples.

Due to Assumption 3, the strata proportions π_c, π_a, and π_n can be point identified as

$$\pi_a = P[D^{obs}_i = 1 | Z_i = 0] \tag{5}$$

$$\pi_n = P[D^{obs}_i = 0 | Z_i = 1]$$

$$\pi_c = 1 - \pi_a - \pi_n.$$

3 Identification results for single binary outcomes

We first derive identification results for the primary outcome Y_1 without imposing ER. Depending on the maintained assumptions, the bounds we derive here and in the following Sections only include the set of parameters values that are consistent with the assumptions, and they are sharp in the sense that they exhaust all information. Sharp bounds are also referred to as the identification region, and we also show that, in our setting, the sharp bounds we derive coincide with maximum likelihood regions (Tamer, 2010). The proof of the following proposition is sketched in Appendix A.

Proposition 1 *Under Assumptions 1, 2 and 3, $P_{c0}^{(1\cdot)}$ and $P_{c1}^{(1\cdot)}$, $P_{n0}^{(1\cdot)}$ and $P_{a1}^{(1\cdot)}$ can be bounded. Detailed expressions are reported in the Appendix. For example, bounds for $P_{c0}^{(1\cdot)}$ are:*

$$L_{P_{c0}^{(1\cdot)}} = \max\left(\frac{P[Y_{i1}^{obs}=1|Z_i=0,D_i^{obs}=0](\pi_c+\pi_n)}{\pi_c} - \frac{\pi_n}{\pi_c}, 0\right) \leq P_{c0}^{(1\cdot)} \leq \min\left(\frac{P[Y_{i1}^{obs}=1|Z_i=0,D_i^{obs}=0](\pi_c+\pi_n)}{\pi_c}, 1\right) = U_{P_{c0}^{(1\cdot)}} \quad (6)$$

Corollary 1 *Under Assumptions 1, 2 and 3, ITT effects can be bounded as:*

$$L_{P_{c1}^{(1\cdot)}} - U_{P_{c0}^{(1\cdot)}} \leq P_{c1}^{(1\cdot)} - P_{c0}^{(1\cdot)} \leq U_{P_{c1}^{(1\cdot)}} - L_{P_{c0}^{(1\cdot)}}, \quad (7)$$

$$P[Y_{i1}^{obs}=1|Z_i=1,D_i^{obs}=0] - U_{P_{n0}^{(1\cdot)}} \leq P_{n1}^{(1\cdot)} - P_{n0}^{(1\cdot)} \leq P[Y_{i1}^{obs}=1|Z_i=1,D_i^{obs}=0] - L_{P_{n0}^{(1\cdot)}}, \quad (8)$$

$$L_{P_{a1}^{(1\cdot)}} - P[Y_{i1}^{obs}=1|Z_i=0,D_i^{obs}=1] \leq P_{a1}^{(1\cdot)} - P_{a0}^{(1\cdot)} \leq U_{P_{a1}^{(1\cdot)}} - P[Y_{i1}^{obs}=1|Z_i=0,D_i^{obs}=1]. \quad (9)$$

As far as the ITT effect for never-takers and always-takers, bounds in Corollary 1 coincide with the bounds on principal *direct* effects in Zhang and Rubin (2003), Imai (2008), Lee (2009) (see also Richardson et al., 2011). Let now focus on the secondary outcome, Y_2, and resume the identification results in the presence of stochastic exclusion restrictions for never-takers and always-takers (Assumption 4). The proof of the following proposition is sketched in Appendix A.

Proposition 2 *Under Assumptions 1, 2, 3 and 4, $P_{c0}^{(\cdot 1)}$, $P_{c1}^{(\cdot 1)}$, $P_{n0}^{(\cdot 1)}$ and $P_{a1}^{(\cdot 1)}$ can be identified as:*

$$P_{n1}^{(\cdot 1)} = P_{n0}^{(\cdot 1)} = P[Y_{i2}^{obs}=1|Z_i=1,D_i^{obs}=0], \quad (10)$$

$$P_{a0}^{(\cdot 1)} = P_{a1}^{(\cdot 1)} = P[Y_{i2}^{obs}=1|Z_i=0,D_i^{obs}=1]. \quad (11)$$

$$P_{c0}^{(\cdot 1)} = \frac{P[Y_{i2}^{obs}=1|Z_i=0,D_i^{obs}=0](\pi_c+\pi_n) - \pi_n P[Y_{i2}^{obs}=1|Z_i=1,D_i^{obs}=0]}{\pi_c}, \quad (12)$$

$$P_{c1}^{(\cdot 1)} = \frac{P[Y_{i2}^{obs}=1|Z_i=1,D_i^{obs}=1](\pi_c+\pi_a) - \pi_a P[Y_{i2}^{obs}=1|Z_i=0,D_i^{obs}=1]}{\pi_c}. \quad (13)$$

The same framework can be used to derive identification results for the distribution of a binary covariate, X, within subpopulations. In this case the stochastic exclusion restriction holds by design, i.e., by the randomization of the instrument, within all three latent subpopulations, so that the distribution of X within subpopulations can be identified using analogous results:

$$P[X_i = 1|Z_i = 1, G_i = a] = P[X_i = 1|Z_i = 0, G_i = a] = P[X_i = 1|Z_i = 0, D_i^{obs} = 1] \tag{14}$$

$$P[X_i = 1|Z_i = 0, G_i = n] = P[X_i = 1|Z_i = 1, G_i = n] = P[X_i = 1|Z_i = 1, D_i^{obs} = 0] \tag{15}$$

$$P[X_i = 1|Z_i = 1, G_i = c] = P[X_i = 1|Z_i = 0, G_i = c] = \frac{P[X_i = 1|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n) - \pi_n P[X_i = 1|Z_i = 1, D_i^{obs} = 0]}{\pi_c}, \tag{16}$$

where the distribution of X for compliers is also equal to $\frac{P[X_i=1|Z_i=1,D_i^{obs}=1](\pi_c+\pi_a)-\pi_a P[X_i=1|Z_i=0,D_i^{obs}=1]}{\pi_c}$.

4 Bivariate binary outcome and partial exclusion restriction

Consider now the bivariate case, with two binary outcomes. Quantities of interest related to the primary outcome can be written as follows:

$$P_{c0}^{(1\cdot)} = P_{c0}^{(11)} + P_{c0}^{(10)},$$

$$P_{c1}^{(1\cdot)} = P_{c1}^{(11)} + P_{c1}^{(10)},$$

$$P_{n0}^{(1\cdot)} = P_{n0}^{(11)} + P_{n0}^{(10)},$$

$$P_{a1}^{(1\cdot)} = P_{a1}^{(11)} + P_{a1}^{(10)}.$$

As a consequence, bounds of these quantities can be obtained by first bounding the joint probabilities and then summing up the bounds. It can be easily shown that, without imposing the exclusion restriction on any of the two outcomes, the same bounds in Proposition 1 and Corollary 1 are obtained. The secondary outcome does not help sharpening the bounds if no exclusion restriction is imposed on it. This is a different result from the parametric case, where the joint modelling of two outcomes usually improves inference (both from a frequentist and a Bayesian perspective) in terms of increased precision and reduced bias, even if no exclusion restriction on the second is imposed (Frumento *et al.*, 2011a; Mattei *et al.*, 2011; see also Jo and Muthen, 2001).

Assume now that the partial stochastic exclusion restriction (Assumption 4) holds. Assumption 4 can be also expressed as follows:

$$P_{a0}^{(\cdot 1)} = P_{a0}^{(11)} + P_{a0}^{(01)} = P_{a1}^{(\cdot 1)} = P_{a1}^{(11)} + P_{a1}^{(01)}, \tag{17}$$

$$P_{n0}^{(\cdot 1)} = P_{n0}^{(11)} + P_{n0}^{(01)} = P_{n1}^{(\cdot 1)} = P_{n1}^{(11)} + P_{n1}^{(01)}.$$

Using the following relationships between joint and marginal probabilities:

$$0 \leq P_{n0}^{(11)} \leq P_{n0}^{(\cdot 1)}, 0 \leq P_{a1}^{(11)} \leq P_{a1}^{(\cdot 1)}, 0 \leq P_{c0}^{(11)} \leq P_{c0}^{(\cdot 1)}, 0 \leq P_{c1}^{(11)} \leq P_{c1}^{(\cdot 1)}, \tag{18}$$

$$0 \leq P_{n0}^{(10)} \leq P_{n0}^{(\cdot 0)}, 0 \leq P_{a1}^{(10)} \leq P_{a1}^{(\cdot 0)}, 0 \leq P_{c0}^{(10)} \leq P_{c0}^{(\cdot 0)}, 0 \leq P_{c1}^{(10)} \leq P_{c1}^{(\cdot 0)}, \tag{19}$$

together with (17), leads to tighter bounds. The intuition is that the joint probabilities are bounded above by marginal probabilities that can be identified due to the partial ER on the secondary outcome. This is formally shown in the proof of the following proposition (see Appendix A).

Proposition 3 *Under Assumptions 1, 2, 3 and 4, $P_{c0}^{(11)}$, $P_{c0}^{(10)}$, $P_{c1}^{(11)}$, $P_{c1}^{(10)}$, $P_{n0}^{(11)}$, $P_{n0}^{(10)}$, $P_{a1}^{(11)}$ and $P_{a1}^{(10)}$ can be bounded. Detailed expressions are reported in the Appendix. For example, bounds for $P_{c0}^{(11)}$ and $P_{c0}^{(10)}$ are:*

$$P_{c0}^{(11)} \geq \max\left(\frac{P[\mathbf{Y}_i^{obs} = 11 | Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_c} - \frac{\pi_n}{\pi_c} P[Y_{i2}^{obs} = 1 | Z_i = 1, D_i^{obs} = 0], 0\right) = L_{P_{c0}^{(11)}} \tag{20}$$

$$P_{c0}^{(11)} \leq \min\left(\frac{P[\mathbf{Y}_i^{obs} = 11 | Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_c}, \frac{P[Y_{i2}^{obs} = 1 | Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n) - \pi_n P[Y_{i2}^{obs} = 1 | Z_i = 1, D_i^{obs} = 0]}{\pi_c}\right) = U_{P_{c0}^{(11)}}$$

$$P_{c0}^{(10)} \geq \max\left(\frac{P[\mathbf{Y}_i^{obs} = 10 | Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_c} - \frac{\pi_n}{\pi_c} P[Y_{i2}^{obs} = 0 | Z_i = 1, D_i^{obs} = 0], 0\right) = L_{P_{c0}^{(10)}} \tag{21}$$

$$P_{c0}^{(10)} \leq \min\left(\frac{P[\mathbf{Y}_i^{obs} = 10 | Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_c}, \frac{P[Y_{i2}^{obs} = 0 | Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n) - \pi_n P[Y_{i2}^{obs} = 0 | Z_i = 1, D_i^{obs} = 0]}{\pi_c}\right) = U_{P_{c0}^{(10)}}$$

These sharp bounds can also be obtained by parameterizing the likelihood of the observed data as a function of the parameters of interest and other nuisance parameters, $\theta = (P_{c0}^{(11)}, P_{c0}^{(10)}, P_{c0}^{(01)}, P_{c1}^{(11)}, P_{c1}^{(10)}, P_{c1}^{(01)}, P_{n0}^{(11)}, P_{n0}^{(10)}, P_{n0}^{(01)}, P_{a1}^{(11)}, P_{a1}^{(10)}, P_{a1}^{(01)}, \pi_a, \pi_n)$, as shown in Appendix A. By maximizing the likelihood under Assumption 4, we can show that the identified set, as the argmax of the likelihood, is a set of parameters values that coincides with the bounds in Proposition 3.

Corollary 2 *Under Assumptions 1, 2, 3 and 4, bounds for $P_{c0}^{(1\cdot)}$, $P_{c1}^{(1\cdot)}$, $P_{n0}^{(1\cdot)}$ and $P_{a1}^{(1\cdot)}$, can be obtained as follows:*

$$\begin{aligned}
L_{P_{c0}^{(11)}} + L_{P_{c0}^{(10)}} \leq P_{c0}^{(1\cdot)} \leq U_{P_{c0}^{(11)}} + U_{P_{c0}^{(10)}}, \\
L_{P_{c1}^{(11)}} + L_{P_{c1}^{(10)}} \leq P_{c1}^{(1\cdot)} \leq U_{P_{c1}^{(11)}} + U_{P_{c1}^{(10)}}. \\
L_{P_{n0}^{(11)}} + L_{P_{n0}^{(10)}} \leq P_{n0}^{(1\cdot)} \leq U_{P_{n0}^{(11)}} + U_{P_{n0}^{(10)}}, \\
L_{P_{a1}^{(11)}} + L_{P_{a1}^{(10)}} \leq P_{a1}^{(1\cdot)} \leq U_{P_{a1}^{(11)}} + U_{P_{a1}^{(10)}}.
\end{aligned} \tag{22}$$

In order to interpret the bounds, take as an example the sum $L_{P_{c0}^{(11)}} + L_{P_{c0}^{(10)}}$; it would correspond to the lower bound obtained in Section 3 if both $L_{P_{c0}^{(11)}}$ and $L_{P_{c0}^{(10)}}$ were greater than zero. The lower bound in (22) becomes strictly greater than the lower bound in (6) if at least one of the two terms is equal to zero. For example, suppose that

$$\frac{P[\mathbf{Y}_i^{obs} = 11|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_c} - \frac{\pi_n}{\pi_c} P[Y_{i2}^{obs} = 1|Z_i = 1, D_i^{obs} = 0] \tag{23}$$

in (20) is > 0 and

$$\frac{P[\mathbf{Y}_i^{obs} = 10|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_c} - \frac{\pi_n}{\pi_c} P[Y_{i2}^{obs} = 0|Z_i = 1, D_i^{obs} = 0] \tag{24}$$

in (21) is < 0; in this case the *new* bound is equal to (20) and it is greater than (6), which is implicitly obtained by adding a negative quantity, (24), to (23). The same is true for the lower bounds of $P_{c1}^{(1\cdot)}$, $P_{n0}^{(1\cdot)}$ and $P_{a1}^{(1\cdot)}$. As for the upper bounds, if both $U_{P_{c0}^{(11)}}$ and $U_{P_{c0}^{(10)}}$ were equal to $U_{P_{c0}^{(11)}} = \frac{P[\mathbf{Y}_i^{obs}=11|Z_i=0,D_i^{obs}=0](\pi_c+\pi_n)}{\pi_c}$ and $U_{P_{c0}^{(10)}} = \frac{P[\mathbf{Y}_i^{obs}=10|Z_i=0,D_i^{obs}=0](\pi_c+\pi_n)}{\pi_c}$, then their sum in (22) would be exactly equal to the upper bound in (6). On the contrary, if either $U_{P_{c0}^{(11)}}$ or $U_{P_{c0}^{(10)}}$ is different from the above quantities, then a strictly smaller upper bound for $P_{c0}^{(1\cdot)}$ is obtained. A similar argument holds for the upper bounds of $P_{c1}^{(1\cdot)}$, $P_{n0}^{(1\cdot)}$ and $P_{a1}^{(1\cdot)}$.

Corollary 3 *Under Assumptions 1, 2, 3 and 4, ITT effects can be bounded as:*

$$\begin{gathered}
L^*_{P_{c1}^{(1\cdot)}} - U^*_{P_{c0}^{(1\cdot)}} \leq P_{c1}^{(1\cdot)} - P_{c0}^{(1\cdot)} \leq U^*_{P_{c1}^{(1\cdot)}} - L^*_{P_{c0}^{(1\cdot)}}, \\
P[Y_{1i}^{obs} = 1|Z_i = 1, D_i^{obs} = 0] - U^*_{P_{n0}^{(1\cdot)}} \leq P_{n1}^{(1\cdot)} - P_{n0}^{(1\cdot)} \leq P[Y_{1i}^{obs} = 1|Z_i = 1, D_i^{obs} = 0] - L^*_{P_{n0}^{(1\cdot)}}, \\
L^*_{P_{a1}^{(1\cdot)}} - P[Y_{1i}^{obs} = 1|Z_i = 0, D_i^{obs} = 1] \leq P_{a1}^{(1\cdot)} - P_{a0}^{(1\cdot)} \leq U^*_{P_{a1}^{(1\cdot)}} - P[Y_{1i}^{obs} = 1|Z_i = 0, D_i^{obs} = 1],
\end{gathered} \tag{25}$$

where

$$\begin{gathered}
L^*_{P_{c0}^{(1\cdot)}} = L_{P_{c0}^{(11)}} + L_{P_{c0}^{(10)}}, \quad U^*_{P_{c0}^{(1\cdot)}} = U_{P_{c0}^{(11)}} + U_{P_{c0}^{(10)}}, \\
L^*_{P_{c1}^{(1\cdot)}} = L_{P_{c1}^{(11)}} + L_{P_{c1}^{(10)}}, \quad U^*_{P_{c1}^{(1\cdot)}} = U_{P_{c1}^{(11)}} + U_{P_{c1}^{(10)}}, \\
L^*_{P_{n0}^{(1\cdot)}} = L_{P_{n0}^{(11)}} + L_{P_{n0}^{(10)}}, \quad U^*_{P_{n0}^{(1\cdot)}} = U_{P_{n0}^{(11)}} + U_{P_{n0}^{(10)}}, \\
L^*_{P_{a1}^{(1\cdot)}} = L_{P_{a1}^{(11)}} + L_{P_{a1}^{(10)}}, \quad U^*_{P_{a1}^{(1\cdot)}} = U_{P_{a1}^{(11)}} + U_{P_{a1}^{(10)}}.
\end{gathered}$$

These bounds can be used to check if data falsify exclusion restriction assumptions on the primary outcome, similarly to what has been done by Huber and Mellace (2011). Note that, because our bounds are tighter than the ones derived from moment inequalities used to prove Proposition 1 (see Appendix A) as in Huber and Mellace (2011), the implied testing procedure will have higher power. They, as well as other authors (e.g., Zhang and Rubin , 2003), usually impose additional restrictions related to the primary outcome distribution of different subpopulations, such as stochastic dominance, to tighten bounds or increase testing power. We instead obtain tighter bounds without imposing any additional assumption on the primary outcome, but only using restrictions following from randomization and exclusion restriction on an auxiliary variable.

4.1 Using a covariate as an auxiliary variable

When using the joint distribution of the primary outcome and a covariate, $[Y_{i1}(z), X_i)]'$, under Assumptions 1, 2 and 3 only, bounds for the quantities in Proposition 2, Corollary 1 and Corollary 2 are obtained simply substituting Y_{i2}^{obs} with X_i in all expressions. In Appendix B we show that our bounds correspond to those that would be obtained by averaging bounds on conditional ITT, $E[Y_{i1}(1) - Y_{i1}(0)|G_i = g, X_i = x]$ for $g = c, n, a$ and $x = 0, 1$, over the distribution of X.

This result deserves some special remarks. First, working with the joint distribution highlights the usefulness of using covariates not only when this is required by the assumptions (e.g., as in Lechner and Melly, 2010; or Frolich, 2006), but in general as a tool to reduce the identified set for partially identified estimands. Second, in finite samples, the benefit of using covariates that we have shown would complement the covariate-adjustment procedures used in randomized studies to increase precision and reduce bias (Imbens and Rubin, 2012). Because more than one covariate can be exploited to bound the same ITT effect, even tighter bounds can be obtained by intersecting bounds derived using different binary covariates. Third, our way of deriving bounds, which makes use of restrictions on the joint distribution, facilitates the use of a secondary outcome as an auxiliary variable: while it would be possible to proceed with a conditional analysis also with a secondary outcome, such conditional analysis would not be straightforward, as it is with a covariate. In fact, conditional analysis would not simply involve stratifying on Y_{i2}^{obs}, but it would involve conditioning on $Y_{i2}(z)$ separately by treatment arm, and then combining results in a non-standard fashion (see Appendix B).

5 Additional restrictions on the auxiliary variable

We have shown that assuming the exclusion restriction for a and n for the secondary outcome or using a covariate helps tightening the bounds. Now, we investigate if additional characteristics of the distribution of the auxiliary variable may tighten the bounds to an even larger extent. The intuition is that, on one hand, the auxiliary variable should help identification the stronger its association is with the compliance status. On the other hand, we expect to sharpen inference also the stronger its association is with the primary outcome.

To support these intuitions, we now consider two limiting cases. The first one is when Y_2 is perfectly associated with the compliance behavior and, specifically, when $Y_2 = \mathcal{I}(G = c)$, where $\mathcal{I}$ represents the indicator function. This implies the following equalities:

$$P_{n1}^{(\cdot 1)} = P_{n0}^{(\cdot 1)} = 0, \quad P_{a1}^{(\cdot 1)} = P_{a0}^{(\cdot 1)} = 0, \quad P_{c1}^{(\cdot 1)} = P_{c0}^{(\cdot 1)} = 1, \tag{26}$$

$$P_{n1}^{(11)} = P_{n0}^{(11)} = 0, \quad P_{a1}^{(11)} = P_{a0}^{(11)} = 0, \quad P_{c1}^{(10)} = P_{c0}^{(10)} = 0. \tag{27}$$

Under these restrictions, bound in Proposition 3 collapse.

Corollary 4 *Under Assumptions 1, 2, 3, 4 and if $Y_2 = \mathcal{I}(G = c)$, bounds for $P_{c0}^{(1\cdot)}$, $P_{c1}^{(1\cdot)}$, $P_{n0}^{(1\cdot)}$ and $P_{a1}^{(1\cdot)}$, and so bounds for ITT effects, collapse as follows:*

$$P_{c0}^{(1\cdot)} = P_{c0}^{(11)} = \frac{P[\mathbf{Y}_i^{obs} = 11|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_c}, \tag{28}$$

$$P_{c1}^{(1\cdot)} = P_{c1}^{(11)} = \frac{P[\mathbf{Y}_i^{obs} = 11|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a)}{\pi_c}, \tag{29}$$

$$P_{n0}^{(1\cdot)} = P_{n0}^{(10)} = \frac{P[\mathbf{Y}_i^{obs} = 10|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_n}, \tag{30}$$

$$P_{a1}^{(1\cdot)} = P_{a1}^{(10)} = \frac{P[\mathbf{Y}_i^{obs} = 10|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a)}{\pi_a}. \tag{31}$$

Bounds collapse if the secondary outcome predicts with no uncertainty the compliance status; this is also true if we use a covariate such that $X = \mathcal{I}(G = c)$. Table 1 shows an example of this limiting case.

TABLE 1- Perfect prediction of compliance status

The second limiting case is when the secondary outcome is perfectly dependent on the primary outcome conditional on the compliance status and the treatment assignment. Specifically, suppose that

$$P^{(11)}_{n0} = P^{(\cdot 1)}_{n0} = P^{(1\cdot)}_{n0}, \quad P^{(11)}_{c0} = P^{(\cdot 1)}_{c0} = P^{(1\cdot)}_{c0}, \tag{32}$$

and

$$P^{(11)}_{a1} = P^{(\cdot 1)}_{a1} = P^{(1\cdot)}_{a1}, \quad P^{(11)}_{c1} = P^{(\cdot 1)}_{c1} = P^{(1\cdot)}_{c1}. \tag{33}$$

Corollary 5 *Under Assumptions 1, 2, 3, 4 and if (32) and (33) hold, bounds for $P^{(1\cdot)}_{c0}$, $P^{(1\cdot)}_{c1}$, $P^{(1\cdot)}_{n0}$ and $P^{(1\cdot)}_{a1}$, and so bounds for ITT effects, collapse as follows:*

$$P^{(1\cdot)}_{c0} = P^{(11)}_{c0} = \frac{P[\mathbf{Y}^{obs}_i = 11|Z_i = 0, D^{obs}_i = 0](\pi_c + \pi_n)}{\pi_c} - \frac{\pi_n}{\pi_c} P[Y^{obs}_{i2} = 1|Z_i = 1, D^{obs}_i = 0], \tag{34}$$

$$P^{(1\cdot)}_{c1} = P^{(11)}_{c1} = \frac{P[\mathbf{Y}^{obs}_i = 11|Z_i = 1, D^{obs}_i = 1](\pi_c + \pi_a)}{\pi_c} - \frac{\pi_a}{\pi_c} P[Y^{obs}_{i2} = 1|Z_i = 0, D^{obs}_i = 1], \tag{35}$$

$$P^{(1\cdot)}_{n0} = P^{(11)}_{n0} = \frac{P[\mathbf{Y}^{obs}_i = 11|Z_i = 0, D^{obs}_i = 0](\pi_c + \pi_n)}{\pi_n} - \frac{\pi_c}{\pi_n} P[Y^{obs}_{i2} = 1|Z_i = 1, D^{obs}_i = 0], \tag{36}$$

$$P^{(1\cdot)}_{a1} = P^{(11)}_{a1} = \frac{P[\mathbf{Y}^{obs}_i = 11|Z_i = 1, D^{obs}_i = 1](\pi_c + \pi_a)}{\pi_a} - \frac{\pi_c}{\pi_a} P[Y^{obs}_{i2} = 1|Z_i = 0, D^{obs}_i = 1]. \tag{37}$$

Note that perfect dependence in the sense of (32) and (33) does not imply the exclusion restriction to hold also for the primary outcome, because $P^{(1\cdot)}_{n0}$ may differ from $P^{(1\cdot)}_{n1}$, and $P^{(1\cdot)}_{a0}$ from $P^{(1\cdot)}_{a1}$. Table 2 shows an example where, given the same marginal distributions, different degrees of association between the two outcomes lead to tighter or less tight bounds [4].

TABLE 2 - Association with the primary outcome

6 Latent independence as an identifying assumption

Our setup may also suggest alternative identifying assumptions of ITT effects. We can show that ITT effects can be point-identified if we assume that the two outcomes are independent conditional

[4] Note that perfect dependence can only be achieved when the two marginal distributions are the same, that is, when the frequencies in one of the two diagonals are zero. This is also the only case where the correlation coefficient (that coincides with the phi-coefficient for contingency tables) may reach its maximum absolute value of 1.

on the compliance status. The identification assumption is a form of *latent independence*, in the sense that independence holds only conditional on a latent variable. This is formalized as follows:

$$P_{gz}^{(11)} = P_{gz}^{(1\cdot)} P_{gz}^{(\cdot 1)}, \quad P_{gz}^{(10)} = P_{gz}^{(1\cdot)} P_{gz}^{(\cdot 0)}, \tag{38}$$

$$P_{gz}^{(01)} = P_{gz}^{(0\cdot)} P_{gz}^{(\cdot 1)}, \quad P_{gz}^{(00)} = P_{gz}^{(0\cdot)} P_{gz}^{(\cdot 0)}, \tag{39}$$

for $g = \{c, n, a\}$ and $z = \{0, 1\}$.

The following proposition is proved in Appendix A.

Proposition 4 *Under Assumptions 1, 2, 3, 4, and (38) and (39), the quantities $P_{c0}^{(1\cdot)}$, $P_{c1}^{(1\cdot)}$, $P_{n0}^{(1\cdot)}$ and $P_{a1}^{(1\cdot)}$ can be point-identified as follows:*

$$P_{c0}^{(1\cdot)} = \frac{\pi_n + \pi_c}{\pi_c} \cdot \left\{ \frac{P[\mathbf{Y}_i^{obs} = 10 | Z_i = 0, D_i^{obs} = 0)] P_{n0}^{(\cdot 1)} - P[\mathbf{Y}_i^{obs} = 11 | Z_i = 0, D_i^{obs} = 0](1 - P_{n0}^{(\cdot 1)})}{P_{n0}^{(\cdot 1)} - P_{c0}^{(\cdot 1)}} \right\}, \tag{40}$$

$$P_{c1}^{(1\cdot)} = \frac{\pi_a + \pi_c}{\pi_c} \cdot \left\{ \frac{P[\mathbf{Y}_i^{obs} = 10 | Z_i = 1, D_i^{obs} = 1)] P_{a1}^{(\cdot 1)} - P[\mathbf{Y}_i^{obs} = 11 | Z_i = 1, D_i^{obs} = 1](1 - P_{a1}^{(\cdot 1)})}{P_{a1}^{(\cdot 1)} - P_{c1}^{(\cdot 1)}} \right\}, \tag{41}$$

$$P_{n0}^{(1\cdot)} = \frac{\pi_n + \pi_c}{\pi_n} \cdot \left\{ \frac{P[\mathbf{Y}_i^{obs} = 11 | Z_i = 0, D_i^{obs} = 0](1 - P_{c0}^{(\cdot 1)}) - P[\mathbf{Y}_i^{obs} = 10 | Z_i = 0, D_i^{obs} = 0)] P_{c0}^{(\cdot 1)}}{P_{n0}^{(\cdot 1)} - P_{c0}^{(\cdot 1)}} \right\}, \tag{42}$$

$$P_{a1}^{(1\cdot)} = \frac{\pi_a + \pi_c}{\pi_a} \cdot \left\{ \frac{P[\mathbf{Y}_i^{obs} = 11 | Z_i = 1, D_i^{obs} = 1](1 - P_{c1}^{(\cdot 1)}) - P[\mathbf{Y}_i^{obs} = 10 | Z_i = 1, D_i^{obs} = 1)] P_{c1}^{(\cdot 1)}}{P_{a1}^{(\cdot 1)} - P_{c1}^{(\cdot 1)}} \right\}. \tag{43}$$

Simple estimators of $P_{c0}^{(1\cdot)}$, $P_{c1}^{(1\cdot)}$, $P_{n0}^{(1\cdot)}$ and $P_{a1}^{(1\cdot)}$ are obtained by substituting the observable distributions with their sample counterparts. Proposition 4 essentially shows that if an outcome or a covariate are found that are latent independent of the primary outcome, they essentially serve as an additional instrument. As the ER, latent independence cannot be directly tested, but does have testable implications and can thus be falsified by the data. For example, if it holds quantities in (40)-(43) should lie in [0, 1].

Results in Proposition 4 have results in Ding et al. (2011) as a special case. In the paper a covariate independent of the primary outcome is used to identify the effect on the subpopulation of the always-survivors. We instead use it to identify effects in all the three subgroups, and extend it to a secondary outcome, thus not restricting to a covariate. Table 3 shows an example where the assumption of latent independence holds. The example shows that bounds do not collapse in this case but remain rather wide; however if latent independence is assumed, results in Proposition 4 can be used to correctly point-identify ITT effects.

TABLE 3 - Latent Independence

7 Illustrative empirical examples

For illustrative purposes, two randomized studies with noncompliance, where the ER of the random assignment has been questioned, are analyzed. In what follows, due to the large sample sizes, we abstract from providing confidence intervals which account of sampling variability, although those could be derived in various ways (Manski and Imbens, 2004).

The first example is the study on influenza vaccinations, previously analyzed by McDonald et al. (1992) and by Hirano et al. (2000). In this study, physicians were randomly selected to receive a letter encouraging them to inoculate patients at risk for flu (Z). The treatment of interest is the actual flu shot (D), and the outcome is an indicator for flu-related hospital visits (Y_1). A standard ITT analysis suggested a moderate effect of assignment. The analysis of Hirano et al. (2000) suggested that there is little evidence that this ITT effect is actually due to the taking of the vaccine. In fact, under a plausible Bayesian model, they found that the subpopulation of the patients who would receive the vaccine regardless of whether their physician received a letter appears to benefit as much from the letter (i.e., from assignment) as the subpopulation of patients who would only receive the vaccine if their physician received the encouragement letter. Their analysis suggested a strong violation of the ER for always-takers. We reanalyzed the data from this study using the 2893 individuals observed in 1980, and report the observed distributions of Y_1 and a set of 8 binary covariates in Table 4: X_1=chronic obstructive pulmonary disease (COPD), X_2=age above median age, X_3=liver disease, X_4=sex, X_5=renal disease, X_6=heart disease, X_7=diabetes, X_8=race. Consistently with the results in Hirano et al. (2000), the bounds on ITT$_a$, even without using auxiliary variables, only cover negative values, highlighting a reduction of hospitalization for the always-vaccinated individuals receiving the letter. All the covariates show a weak association with the primary outcome, so that we do not expect major improvements of the bounds on ITT$_c$, ITT$_n$, and ITT$_a$. Bounds reported in Table 4 are derived as intersection of the bounds obtained with each single covariate, and lead to bounds with slightly smaller width. In this specific study, our analysis shows that covariates, used in the Bayesian analysis conducted by Hirano et al. (2000), have contributed to reduce the posterior variability around the identified set, but their contribution to the reduction of the size of the identified set is negligible.

TABLE 4 - Influenza Vaccine encouragement study

The second example is the National Job Corps (JC) Study, a randomized experiment performed in the mid-1990s to evaluate the effects of participation in JC (D), a large job training program for economically disadvantaged youths aged 16 to 24 years. A random sample of eligible applicants ($N = 13987$) was randomly assigned into treatment and control groups (Z), with the second group being denied access to JC for three years. Both groups were tracked at baseline, soon and at 12, 30 and 48 months after randomization. Previous works have concentrated of global ITT effects, i.e., effects of being assigned to enroll in Job Corps (e.g., Lee, 2009; Zhang et al., 2009). However, non-compliance was present, as only 68% of those assigned to the treatment group actually enrolled in JC within 6 months from assignment. When estimating the effect on compliers, the ER for never-takers was always maintained (e.g., Frumento et al., 2011b). However being denied enrollment in JC, as opposed to deciding not to accept the offer to enroll, may, in principle, affect the labor market behavior of never-takers, especially in the short-term. For example, the denial may encourage applicants to temporarily look for alternative forms of training, possibly reducing their job search intensity. However, because they are people who are not willing to be trained when offered the opportunity, the overall amount of training that never-takers are expected to get is plausibly the same irrespective of initial assignment. As a consequence, assignment should not have any effect on long-term employment, so that the ER is more plausible for long-term labor outcomes.

To limit exposition, here we concentrate only on short-term (12-months after randomization) effects on employment (Y_1) and use the log-term employment indicator (at week 130 after randomization) as a secondary outcome (Y_2); we use only observations where both outcomes and the treatment indicator are not missing (N=13193). Data and results are reported in Table 5; the two observed outcomes are strongly associated when $Z = 0$, so that our bounds are expected to be a lot tighter than the ones derived without using the secondary outcome. Table 5 highlights this, although we found no evidence of the violation of the ER for never-takers, because the bounds on ITT_n are narrower but still cover 0. On the other hand, bounds for ITT_c are narrower and point to a negative effect on employment for compliers of at least 1% points, confirming lock-in effects of those participating in the program (van Ours, 2004; Lechner and Wunsch, 2009; Frumento et al., 2011b).

TABLE 5 - Job Corps study

8 Concluding remarks

We used restrictions on the joint distribution of a primary outcome and an auxiliary variable (a secondary outcome or a covariate) to derive nonparametric bounds for intention-to-treat effects on the primary outcome on the subpopulations defined by compliance behavior, without requiring the exclusion restriction of the instrument. Bounds available in the literature do not satisfy all the restrictions implied by random assignment (Assumption 1) (and by the partial ER, Assumption 4).

There are four main benefits of the approach we propose.

First, ITT effects only involve outcomes that can potentially be observed. This has clear advantages. For example, the LATE estimand defined in Flores and Flores-Lagunes (2010) involves an *a priori counterfactual* quantity, namely the outcome for compliers when they are assigned to take the treatment and do not take it. Because compliers *do* take the treatment if assigned to take it, data contain no information on this outcome. In this regard, we make a clear distinction of what can be learnt from the data regarding potentially observable quantities, and what can be extrapolated on a priori counterfactuals using additional assumptions.

Second, we argue that there is much to be learnt from bounds on ITT effects. On one hand, ITT effects for noncompliers provide information on the extent of the violation of ERs: the sign of the violation is sometimes identified, and separately for never-takers and always-takers, as shown in our artifical and real-data examples. By sharpening information on the magnitude of the violation, our bounds may provide information on the appropriateness of using methods which allow for locally misspecified instruments (Hausman and Hahn, 2005; Conley et al., 2008, 2011; Nevo and Rosen, 2011). On the other hand, ITT effects for compliers provide information on the possible extent of the effect of the treatment, particularly when compared with ITT effects for noncompliers. For example, assuming that the effects of the treatment and of assignment are additive and that the effect of assignment for noncompliers is the same as the effect of assignment for compliers, then the effect of treatment for compliers can also be bounded.

Third, our setup provides guidelines on which auxiliary variables should be collected and jointly analyzed. Specifically, the stronger the association of an auxiliary variable with the compliance status and/or the primary outcome the narrower the bounds. In this regard, the fact that we proved that information of a secondary outcome, and not only of a covariate, tightens the bounds is particularly useful, because some secondary outcomes are expected to be highly associated with the primary outcome and compliance status.

Fourth, our setup provides alternative point-identifying assumptions, in the form of latent independences, and estimators of ITTs effects under this assumption.

The approach we followed was nonparametric. Note however that in the binary outcome case the parametric/nonparametric distinction is redundant. In this regard, the moment equalities that we used to derive bounds coincide with first order conditions for deriving MLE, so that the derived bounds identify also the maximum likelihood regions. This means that in finite samples, covariates and secondary outcomes can not only be used to increase precision but also to reduce the identified set of weakly identified models.

As a general message, the paper stresses the importance of taking account and exploiting restrictions implied by the randomization that involve other variables related to the one of primary interest. This may prove to be useful also in other settings of broken randomized experiments, other than settings with noncompliance, where typically some (local) causal estimands of interest can only be partially identified.

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Appendix A

Proof of Proposition 1 Under Assumptions 1, 2 and 3, the four observable distributions are equal to:

$$
\begin{aligned}
P[Y_{i1}^{obs} = 1 | Z_i = 0, D_i^{obs} = 1] &= P_{a0}^{(1\cdot)}, \\
P[Y_{i1}^{obs} = 1 | Z_i = 1, D_i^{obs} = 0] &= P_{n1}^{(1\cdot)}, \\
P[Y_{i1}^{obs} = 1 | Z_i = 0, D_i^{obs} = 0] &= \frac{\pi_c P_{c0}^{(1\cdot)} + \pi_n P_{n0}^{(1\cdot)}}{\pi_c + \pi_n}, \\
P[Y_{i1}^{obs} = 1 | Z_i = 1, D_i^{obs} = 1] &= \frac{\pi_c P_{c1}^{(1\cdot)} + \pi_a P_{a1}^{(1\cdot)}}{\pi_c + \pi_a}.
\end{aligned}
\tag{44}
$$

Given that $0 \le P^{(1\cdot)}_{c0}, P^{(1\cdot)}_{c1}, P^{(1\cdot)}_{n0}, P^{(1\cdot)}_{a1} \le 1$, worst case bounds are derived. For example, the lower (upper) bound for $P^{(1\cdot)}_{c0}$ is obtained as the maximum (minimum) of 0 (1) and using (44) when $P^{(1\cdot)}_{n0} = 1$ ($P^{(1\cdot)}_{n0} = 0$):

$$L_{P^{(1\cdot)}_{c0}} = \max\left(\frac{P[Y^{obs}_{i1} = 1|Z_i = 0, D^{obs}_i = 0](\pi_c + \pi_n)}{\pi_c} - \frac{\pi_n}{\pi_c}, 0\right) \le P^{(1\cdot)}_{c0} \le \min\left(\frac{P[Y^{obs}_{i1} = 1|Z_i = 0, D^{obs}_i = 0](\pi_c + \pi_n)}{\pi_c}, 1\right) = U_{P^{(1\cdot)}_{c0}}$$

Analogously, the following bounds are derived:

$$L_{P^{(1\cdot)}_{c1}} = \max\left(\frac{P[Y^{obs}_{i1} = 1|Z_i = 1, D^{obs}_i = 1](\pi_c + \pi_a)}{\pi_c} - \frac{\pi_a}{\pi_c}, 0\right) \le P^{(1\cdot)}_{c1} \le \min\left(\frac{P[Y^{obs}_{i1} = 1|Z_i = 1, D^{obs}_i = 1](\pi_c + \pi_a)}{\pi_c}, 1\right) = U_{P^{(1\cdot)}_{c1}}$$

$$L_{P^{(1\cdot)}_{n0}} = \max\left(\frac{P[Y^{obs}_{i1} = 1|Z_i = 0, D^{obs}_i = 0](\pi_c + \pi_n)}{\pi_n} - \frac{\pi_c}{\pi_n}, 0\right) \le P^{(1\cdot)}_{n0} \le \min\left(\frac{P[Y^{obs}_{i1} = 1|Z_i = 0, D^{obs}_i = 0](\pi_c + \pi_n)}{\pi_n}, 1\right) = U_{P^{(1\cdot)}_{n0}}$$

$$L_{P^{(1\cdot)}_{a1}} = \max\left(\frac{P[Y^{obs}_{i1} = 1|Z_i = 1, D^{obs}_i = 1](\pi_c + \pi_a)}{\pi_a} - \frac{\pi_c}{\pi_a}, 0\right) \le P^{(1\cdot)}_{a1} \le \min\left(\frac{P[Y^{obs}_{i1} = 1|Z_i = 1, D^{obs}_i = 1](\pi_c + \pi_a)}{\pi_a}, 1\right) = U_{P^{(1\cdot)}_{a1}}$$

Proof of Proposition 2 Under Assumptions 1, 2, 3 and 4, the four observable distributions are equal to:

$$P[Y^{obs}_{i2} = 1|Z_i = 0, D^{obs}_i = 1] = P^{(\cdot 1)}_{a0},$$

$$P[Y^{obs}_{i2} = 1|Z_i = 1, D^{obs}_i = 0] = P^{(\cdot 1)}_{n1},$$

$$P[Y^{obs}_{i2} = 1|Z_i = 0, D^{obs}_i = 0] = \frac{\pi_c P^{(\cdot 1)}_{c0} + \pi_n P^{(\cdot 1)}_{n0}}{\pi_c + \pi_n} = \frac{\pi_c P^{(\cdot 1)}_{c0} + \pi_n P^{(\cdot 1)}_{n1}}{\pi_c + \pi_n}, \tag{45}$$

$$P[Y^{obs}_{i2} = 1|Z_i = 1, D^{obs}_i = 1] = \frac{\pi_c P^{(\cdot 1)}_{c1} + \pi_a P^{(\cdot 1)}_{a1}}{\pi_c + \pi_a} = \frac{\pi_c P^{(\cdot 1)}_{c1} + \pi_a P^{(\cdot 1)}_{a0}}{\pi_c + \pi_a}, \tag{46}$$

where the second equalities in (45) and in (46) are due to the exclusion restrictions, so that the system can be univocally solved also in $P^{(\cdot 1)}_{c0}$ and $P^{(\cdot 1)}_{c1}$ as

$$P^{(\cdot 1)}_{c0} = \frac{P[Y^{obs}_{i2} = 1|Z_i = 0, D^{obs}_i = 0](\pi_c + \pi_n) - \pi_n P[Y^{obs}_{i2} = 1|Z_i = 1, D^{obs}_i = 0]}{\pi_c},$$

$$P^{(\cdot 1)}_{c1} = \frac{P[Y^{obs}_{i2} = 1|Z_i = 1, D^{obs}_i = 1](\pi_c + \pi_a) - \pi_a P[Y^{obs}_{i2} = 1|Z_i = 0, D^{obs}_i = 1]}{\pi_c}.$$

Proof of Proposition 3 In order to bound $P^{(11)}_{c0}$, $P^{(10)}_{c0}$, $P^{(11)}_{c1}$, $P^{(10)}_{c1}$, $P^{(11)}_{n0}$, $P^{(10)}_{n0}$, $P^{(11)}_{a1}$ and $P^{(10)}_{a1}$, the relevant observable joint distributions are equal to the following:

$$P[\mathbf{Y}^{obs}_i = 11|Z_i = 0, D^{obs}_i = 0] = \frac{\pi_c P^{(11)}_{c0} + \pi_n P^{(11)}_{n0}}{\pi_c + \pi_n}, \tag{47}$$

$$P[\mathbf{Y}^{obs}_i = 11|Z_i = 1, D^{obs}_i = 1] = \frac{\pi_c P^{(11)}_{c1} + \pi_a P^{(11)}_{a1}}{\pi_c + \pi_a},$$

$$P[\mathbf{Y}^{obs}_i = 10|Z_i = 0, D^{obs}_i = 0] = \frac{\pi_c P^{(10)}_{c0} + \pi_n P^{(10)}_{n0}}{\pi_c + \pi_n},$$

$$P[\mathbf{Y}^{obs}_i = 10|Z_i = 1, D^{obs}_i = 1] = \frac{\pi_c P^{(10)}_{c1} + \pi_a P^{(10)}_{a1}}{\pi_c + \pi_a}.$$

Also, the following inequalities follow from the usual relationship between joint and marginal distributions:

$$0 \le P^{(11)}_{n0} \le P^{(\cdot 1)}_{n0} = P[Y^{obs}_{i2} = 1|Z_i = 1, D^{obs}_i = 0], \tag{48}$$

$$0 \le P^{(11)}_{c0} \le P^{(\cdot 1)}_{c0} =$$

$$= \frac{P[Y_{i2}^{obs} = 1|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n) - \pi_n P[Y_{i2}^{obs} = 1|Z_i = 1, D_i^{obs} = 0]}{\pi_c},$$

$$0 \le P_{a1}^{(11)} \le P_{a1}^{(\cdot 1)} = P[Y_{i2}^{obs} = 1|Z_i = 0, D_i^{obs} = 1],$$

$$0 \le P_{c1}^{(11)} \le P_{c1}^{(\cdot 1)} =$$

$$\frac{P[Y_{i2}^{obs} = 1|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a) - \pi_a P[Y_{i2}^{obs} = 1|Z_i = 0, D_i^{obs} = 1]}{\pi_c},$$

where the equalities follow from results in Proposition 2. Under these restrictions bounds are obtained by using the equalities in (47) and substituting the maximum and minimum values of relevant quantities in (48):

$$P_{c0}^{(11)} \ge \max\left(\frac{P[\mathbf{Y}_i^{obs} = 11|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_c} - \frac{\pi_n}{\pi_c} P[Y_{i2}^{obs} = 1|Z_i = 1, D_i^{obs} = 0], 0\right) = L_{P_{c0}^{(11)}}$$

$$P_{c0}^{(11)} \le \min\left(\frac{P[\mathbf{Y}_i^{obs} = 11|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_c}, \frac{P[Y_{i2}^{obs} = 1|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n) - \pi_n P[Y_{i2}^{obs} = 1|Z_i = 1, D_i^{obs} = 0]}{\pi_c}\right) = U_{P_{c0}^{(11)}}$$

$$P_{c0}^{(10)} \ge \max\left(\frac{P[\mathbf{Y}_i^{obs} = 10|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_c} - \frac{\pi_n}{\pi_c} P[Y_{i2}^{obs} = 0|Z_i = 1, D_i^{obs} = 0], 0\right) = L_{P_{c0}^{(10)}}$$

$$P_{c0}^{(10)} \le \min\left(\frac{P[\mathbf{Y}_i^{obs} = 10|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_c}, \frac{P[Y_{i2}^{obs} = 0|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n) - \pi_n P[Y_{i2}^{obs} = 0|Z_i = 1, D_i^{obs} = 0]}{\pi_c}\right) = U_{P_{c0}^{(10)}}$$

$$P_{c1}^{(11)} \ge \max\left(\frac{P[\mathbf{Y}_i^{obs} = 11|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a)}{\pi_c} - \frac{\pi_a}{\pi_c} P[Y_{i2}^{obs} = 1|Z_i = 0, D_i^{obs} = 1], 0\right) = L_{P_{c1}^{(11)}}$$

$$P_{c1}^{(11)} \le \min\left(\frac{P[\mathbf{Y}_i^{obs} = 11|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a)}{\pi_c}, \frac{P[Y_{i2}^{obs} = 1|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a) - \pi_a P[Y_{i2}^{obs} = 1|Z_i = 0, D_i^{obs} = 1]}{\pi_c}\right) = U_{P_{c1}^{(11)}}$$

$$P_{c1}^{(10)} \ge \max\left(\frac{P[\mathbf{Y}_i^{obs} = 10|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a)}{\pi_c} - \frac{\pi_a}{\pi_c} P[Y_{i2}^{obs} = 0|Z_i = 0, D_i^{obs} = 1], 0\right) = L_{P_{c1}^{(10)}}$$

$$P_{c1}^{(10)} \le \min\left(\frac{P[\mathbf{Y}_i^{obs} = 10|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a)}{\pi_c}, \frac{P[Y_{i2}^{obs} = 0|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a) - \pi_a P[Y_{i2}^{obs} = 0|Z_i = 0, D_i^{obs} = 1]}{\pi_c}\right) = U_{P_{c1}^{(10)}}.$$

$$P_{n0}^{(11)} \ge \max\left(\frac{P[Y_i^{obs} = 11|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_n} - \frac{\pi_c}{\pi_n}\frac{P[\mathbf{Y}_{i2}^{obs} = 1|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n) - \pi_n P[Y_{i2}^{obs} = 1|Z_i = 1, D_i^{obs} = 0]}{\pi_c}, 0\right) = L_{P_{n0}^{(11)}}$$

$$P_{n0}^{(11)} \le \min\left(\frac{P[\mathbf{Y}_i^{obs} = 11|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_n}, P[Y_{i2}^{obs} = 1|Z_i = 1, D_i^{obs} = 0]\right) = U_{P_{n0}^{(11)}}.$$

$$P_{n0}^{(10)} \ge \max\left(\frac{P[\mathbf{Y}_i^{obs} = 10|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_n} - \frac{\pi_c}{\pi_n}\frac{P[Y_{i2}^{obs} = 0|Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n) - \pi_n P[Y_{i2}^{obs} = 0|Z_i = 1, D_i^{obs} = 0]}{\pi_c}, 0\right) = L_{P_{n0}^{(10)}}$$

$$P_{n0}^{(10)} \le \min\left(\frac{P[\mathbf{Y}_i^{obs} = 10|Z_i = 0, P[Y_{i2}^{obs} = 0|Z_i = 1, D_i^{obs} = 0](\pi_c + \pi_n)}{\pi_n}, P[Y_{i2}^{obs} = 0|Z_i = 1, D_i^{obs} = 0]\right) = U_{P_{n0}^{(10)}}$$

$$P_{a1}^{(11)} \ge \max\left(\frac{P[\mathbf{Y}_i^{obs} = 11|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a)}{\pi_a} - \frac{\pi_c}{\pi_a}\frac{P[Y_{i2}^{obs} = 1|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a) - \pi_a P[Y_{i2}^{obs} = 1|Z_i = 0, D_i^{obs} = 1]}{\pi_c}, 0\right) = L_{P_{a1}^{(11)}}$$

$$P_{a1}^{(11)} \le \min\left(\frac{P[\mathbf{Y}_i^{obs} = 11|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a)}{\pi_a}, P[Y_{i2}^{obs} = 1|Z_i = 0, D_i^{obs} = 1]\right) = U_{P_{a1}^{(11)}}$$

$$P_{a1}^{(10)} \ge \max\left(\frac{P[\mathbf{Y}_i^{obs} = 10|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a)}{\pi_a} - \frac{\pi_c}{\pi_a}\frac{P[Y_{i2}^{obs} = 0|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a) - \pi_a P[Y_{i2}^{obs} = 0|Z_i = 0, D_i^{obs} = 1]}{\pi_c}, 0\right) = L_{P_{a1}^{(10)}}$$

$$P_{a1}^{(10)} \le \min\left(\frac{P[\mathbf{Y}_i^{obs} = 10|Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a)}{\pi_a}, P[Y_{i2}^{obs} = 0|Z_i = 0, D_i^{obs} = 1]\right) = U_{P_{a1}^{(10)}}.$$

Bounds as maximum likelihood regions The sharp sets obtained in Proposition 3 can also be derived by parameterizing the likelihood of the observed data as a function of the parameters of interest and other nuisance parameters $\theta = (P^{(11)}_{c0}, P^{(10)}_{c0}, P^{(01)}_{c0}, P^{(11)}_{c1}, P^{(10)}_{c1}, P^{(01)}_{c1}, P^{(11)}_{n0}, P^{(10)}_{n0}, P^{(01)}_{n0}, P^{(11)}_{a1}, P^{(10)}_{a1}, P^{(01)}_{a1}, \pi_a, \pi_n)$. Denote with $\pi_c = 1 - \pi_a - \pi_n$, and $P^{(00)}_{c0} = 1 - P^{(11)}_{c0} - P^{(10)}_{c0} - P^{(01)}_{c0}$, $P^{(00)}_{c1} = 1 - P^{(11)}_{c1} - P^{(10)}_{c1} - P^{(01)}_{c1}$, $P^{(00)}_{n0} = 1 - P^{(11)}_{n0} - P^{(10)}_{n0} - P^{(01)}_{n0}$, $P^{(00)}_{a1} = 1 - P^{(11)}_{a1} - P^{(10)}_{a1} - P^{(01)}_{a1}$. The log-likelihood is:

$$
\begin{aligned}
l(\theta) = E\Big[& Z_i D_i^{obs} \mathcal{I}(\mathbf{Y}_i^{obs} = 11)\log(\pi_c P^{(11)}_{c1} + \pi_a P^{(11)}_{a1}) + Z_i D_i^{obs} \mathcal{I}(\mathbf{Y}_i^{obs} = 10)\log(\pi_c P^{(10)}_{c1} + \pi_a P^{(10)}_{a1}) + \\
& Z_i D_i^{obs} \mathcal{I}(\mathbf{Y}_i^{obs} = 01)\log(\pi_c P^{(01)}_{c1} + \pi_a P^{(01)}_{a1}) + Z_i D_i^{obs} \mathcal{I}(\mathbf{Y}_i^{obs} = 00)\log(\pi_c P^{(00)}_{c1} + \pi_a P^{(00)}_{a1}) + \\
& (1 - Z_i) D_i^{obs} \mathcal{I}(\mathbf{Y}_i^{obs} = 11)\log(\pi_a P^{(11)}_{a0}) + (1 - Z_i) D_i^{obs} \mathcal{I}(\mathbf{Y}_i^{obs} = 10)\log(\pi_a P^{(10)}_{a0}) + \\
& (1 - Z_i) D_i^{obs} \mathcal{I}(\mathbf{Y}_i^{obs} = 01)\log(\pi_a P^{(01)}_{a0}) + (1 - Z_i) D_i^{obs} \mathcal{I}(\mathbf{Y}_i^{obs} = 00)\log(\pi_a P^{(00)}_{a0}) + \\
& Z_i (1 - D_i^{obs}) \mathcal{I}(\mathbf{Y}_i^{obs} = 11)\log(\pi_n P^{(11)}_{n1}) + Z_i (1 - D_i^{obs}) \mathcal{I}(\mathbf{Y}_i^{obs} = 10)\log(\pi_n P^{(10)}_{n1}) + \\
& Z_i (1 - D_i^{obs}) \mathcal{I}(\mathbf{Y}_i^{obs} = 01)\log(\pi_n P^{(01)}_{n1}) + Z_i (1 - D_i^{obs}) \mathcal{I}(\mathbf{Y}_i^{obs} = 00)\log(\pi_n P^{(00)}_{n1}) + \\
& (1 - Z_i)(1 - D_i^{obs}) \mathcal{I}(\mathbf{Y}_i^{obs} = 11)\log(\pi_c P^{(11)}_{c0} + \pi_n P^{(11)}_{n0}) + (1 - Z_i)(1 - D_i^{obs}) \mathcal{I}(\mathbf{Y}_i^{obs} = 10)\log(\pi_c P^{(10)}_{c0} + \pi_n P^{(10)}_{n0}) + \\
& (1 - Z_i)(1 - D_i^{obs}) \mathcal{I}(\mathbf{Y}_i^{obs} = 01)\log(\pi_c P^{(01)}_{c0} + \pi_n P^{(01)}_{n0}) + (1 - Z_i)(1 - D_i^{obs}) \mathcal{I}(\mathbf{Y}_i^{obs} = 00)\log(\pi_c P^{(00)}_{c0} + \pi_n P^{(00)}_{n0})\Big]
\end{aligned}
\tag{49}
$$

Maximizing the above likelihood under Assumption 4 (that is, under restrictions in (17)) we can obtain the argmax as the set of parameters that satisfy (5) and (47). It can be easily shown that due to Assumption 4, the following sums of parameters $P^{(11)}_{a0} + P^{(01)}_{a0} = P^{(11)}_{a1} + P^{(01)}_{a1}$, $P^{(11)}_{n0} + P^{(01)}_{n0} = P^{(11)}_{n1} + P^{(01)}_{n1}$, $P^{(11)}_{c0} + P^{(01)}_{c0}$, and $P^{(11)}_{c1} + P^{(01)}_{c1}$ have unique MLE, that is, they are point identified as in Proposition 2. The maximum likelihood regions for the parameters representing the joint probabilities coincide with the bounds in Proposition 3.

Proof of Proposition 4 Substituting (38) and (39) in the equalities in (47), we have the following system of four equations:

$$
\begin{aligned}
P[\mathbf{Y}_i^{obs} = 11 | Z_i = 0, D_i^{obs} = 0](\pi_c + \pi_n) &= \pi_c P^{(1\cdot)}_{c0} P^{(\cdot 1)}_{c0} + \pi_n P^{(1\cdot)}_{n0} P^{(\cdot 1)}_{n0}, \\
P[\mathbf{Y}_i^{obs} = 11 | Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a) &= \pi_c P^{(1\cdot)}_{c1} P^{(\cdot 1)}_{c1} + \pi_a P^{(1\cdot)}_{a1} P^{(\cdot 1)}_{a1}, \\
P[\mathbf{Y}_i^{obs} = 10 | Z_i = 0, D_i^{obs} = 0)](\pi_n + \pi_c) &= \pi_c P^{(1\cdot)}_{c0}(1 - P^{(\cdot 1)}_{c0}) + \pi_n P^{(1\cdot)}_{n0}(1 - P^{(\cdot 1)}_{n0}), \\
P[\mathbf{Y}_i^{obs} = 10 | Z_i = 1, D_i^{obs} = 1](\pi_c + \pi_a) &= \pi_c P^{(1\cdot)}_{c1}(1 - P^{(\cdot 1)}_{c1}) + \pi_a P^{(1\cdot)}_{a1}(1 - P^{(\cdot 1)}_{a1}).
\end{aligned}
\tag{50}
$$

Now, $P^{(\cdot 1)}_{c0}$, $P^{(\cdot 1)}_{n0}$, $P^{(\cdot 1)}_{c1}$ and $P^{(\cdot 1)}_{a1}$ are identified (see Proposition 2) so that the linear system (50) has only four unknowns $P^{(1\cdot)}_{c0}$, $P^{(1\cdot)}_{c1}$, $P^{(1\cdot)}_{n0}$, and $P^{(1\cdot)}_{a1}$, and can be solved, giving results in (40)-(43).

Appendix B

Deriving bounds by conditioning on X. Under Assumption 1, 2 and 3 we show that, when the auxiliary variable is a binary covariate X, bounds in Corollary 2 can be obtained also with a conditional analysis. To this end, write for example $P^{(1\cdot)}_{c0}$ as

$$
\begin{aligned}
P^{(1\cdot)}_{c0} &= P[Y_{i1}(0) | G_i = c, X_i = 0] \cdot P[X_i = 0 | G_i = c] + P[Y_{i1}(0) | G_i = c, X_i = 1] \cdot P[X_i = 1 | G_i = c] \\
&= P^{(1|0)}_{c0} \cdot P[X_i = 0 | G_i = c] + P^{(1|1)}_{c0} \cdot P[X_i = 1 | G_i = c].
\end{aligned}
\tag{51}
$$

Let introduce the following additional notation: $P[G_i = g|X_i = x] = \pi_{g|x}, \quad g = c, n, a; \quad x = 0, 1$. These conditional strata proportions are point identified as

$$\pi_{a|1} = P[D_i^{obs} = 1|Z_i = 0, X_i = 1],$$

$$\pi_{n|1} = P[D_i^{obs} = 0|Z_i = 1, X_i = 1],$$

$$\pi_{c|1} = 1 - \pi_{a|1} - \pi_{n|1}.$$

This identification result follows from Assumption 1, because $P[G_i = a|Z_i = 0, X_i = 1] = P[G_i = a|Z_i = 1, X_i = 1] = P[G_i = a|X_i = 1]$. Bounds for the conditional quantities, $P_{c0}^{(1|1)}$ and $P_{c0}^{(0|1)}$ in (51), can be obtained applying results in Proposition 1:

$$P_{c0}^{(1|1)} \geq \max\left(\frac{P[Y_{i1}^{obs} = 1|Z_i = 0, D_i^{obs} = 0, X_i = 1](\pi_{c|1} + \pi_{n|1})}{\pi_{c|1}} - \frac{\pi_{n|1}}{\pi_{c|1}}, 0\right) = L_{P_{c0}^{(1|1)}},$$

$$P_{c0}^{(1|1)} \leq \min\left(\frac{P[Y_{i1}^{obs} = 1|Z_i = 0, D_i^{obs} = 0, X_i = 1](\pi_{c|1} + \pi_{n|1})}{\pi_{c|1}}, 1\right) = U_{P_{c0}^{(1|1)}},$$

$$P_{c0}^{(1|0)} \geq \max\left(\frac{P[Y_{i1}^{obs} = 1|Z_i = 0, D_i^{obs} = 0, X_i = 0](\pi_{c|0} + \pi_{n|0})}{\pi_{c|0}} - \frac{\pi_{n|0}}{\pi_{c|0}}, 0\right) = L_{P_{c0}^{(1|0)}},$$

$$P_{c0}^{(1|0)} \leq \min\left(\frac{P[Y_{i1}^{obs} = 1|Z_i = 0, D_i^{obs} = 0, X_i = 0](\pi_{c|0} + \pi_{n|0})}{\pi_{c|0}}, 1\right) = U_{P_{c0}^{(1|0)}}.$$

In order to obtain lower and upper bounds for $P_{c0}^{(1\cdot)}$, $L_{P_{c0}^{(1|1)}}$ and $L_{P_{c0}^{(1|0)}}$, as well as $U_{P_{c0}^{(1|1)}}$ and $U_{P_{c0}^{(1|0)}}$, must be weighted by $P[X_i = 1|G_i = c]$ and $P[X_i = 0|G_i = c]$, respectively, and summed. $P[X_i = 1|G_i = c]$ and $P[X_i = 0|G_i = c]$ are identified as in (15). As an example, denote $\pi_{1|c} = P[X_i = 1|G_i = c]$ and $\pi_1 = P[X_i = 1]$; by weighting $\frac{P[Y_{i1}^{obs}=1|Z_i=0,D_i^{obs}=0,X_i=1](\pi_{c|1}+\pi_{n|1})}{\pi_{c|1}} - \frac{\pi_{n|1}}{\pi_{c|1}}$ in $L_{P_{c0}^{(1|1)}}$, we obtain the following

$$\frac{P[Y_{i1}^{obs} = 1|Z_i = 0, D_i^{obs} = 0, X_i = 1](\pi_{c|1} + \pi_{n|1})}{\pi_{c|1}} \cdot \pi_{1|c} - \frac{\pi_{n|1}}{\pi_{c|1}}\pi_{1|c} =$$

$$\frac{P[Y_{i1}^{obs} = 1|Z_i = 0, D_i^{obs} = 0, X_i = 1](\pi_{c|1} + \pi_{n|1})}{\pi_{c|1}} \cdot \left(\frac{\pi_1 \pi_{c|1}}{\pi_c}\right) - \frac{\pi_{n|1}}{\pi_{c|1}}\left(\frac{\pi_1 \pi_{c|1}}{\pi_c}\right) =$$

$$P[Y_{i1}^{obs} = 1|Z_i = 0, D_i^{obs} = 0, X_i = 1](\pi_{c|1} + \pi_{n|1}) \cdot \frac{\pi_1}{\pi_c} - \pi_{n|1}\frac{\pi_1}{\pi_c} =$$

$$\frac{P[Y_{i1}^{obs} = 1, D_i^{obs} = 0, X_i = 1|Z_i = 0]}{\pi_c} - \frac{\pi_n}{\pi_c}P[X_i = 1|Z_i = 1, D_i^{obs} = 0],$$

where the last equality follows from

$$(\pi_{c|1} + \pi_{n|1}) \cdot \pi_1 = P[G_i = c, X_i = 1] + P[G_i = n, X_i = 1] = P[D_i^{obs} = 0, X_i = 1|Z_i = 0]$$

and

$$\pi_{n|1} \cdot \pi_1 = \pi_n \cdot \pi_{1|n} = \pi_n P[X_i = 1|Z_i = 1, D_i^{obs} = 0].$$

Analogous result can be obtained weighting $\frac{P[Y_{i1}^{obs}=1|Z_i=0,D_i^{obs}=0,X_i=0](\pi_{c|0}+\pi_{n|0})}{\pi_{c|0}} - \frac{\pi_{n|0}}{\pi_{c|0}}$ in $L_{P_{c0}^{(1|0)}}$. Consider now the weighted terms:

$$a^* = \frac{P[Y_{i1}^{obs} = 1, D_i^{obs} = 0, X_i = 1|Z_i = 0]}{\pi_c} - \frac{\pi_n}{\pi_c}P[X_i = 1|Z_i = 1, D_i^{obs} = 0],$$

$$b^* = \frac{P[Y_{i1}^{obs} = 1, D_i^{obs} = 0, X_i = 0|Z_i = 0]}{\pi_c} - \frac{\pi_n}{\pi_c}P[X_i = 0|Z_i = 1, D_i^{obs} = 0],$$

and define $a = \max(0, a^*)$ and $b = \max(0, b^*)$. The lower bound for $P^{(1\cdot)}_{c0}$, $L_{P^{(1\cdot)}_{c0}}$, is obtained as $(a + b)$. If $a = a^*$ and $b = b^*$, then $L_{P^{(1\cdot)}_{c0}}$ is the same lower bound in (6), derived using information only on the primary outcome. If $a^* < 0$ or $b^* < 0$ then $L_{P^{(1\cdot)}_{c0}}$ is equal to the lower bound obtained in Corollary 2.. For example, if $a^* < 0$ and $b^* > 0$, then $L_{P^{(1\cdot)}_{c0}} = b^*$, and can be rewritten as

$$\frac{P[Y^{obs}_{i1} = 1, X_i = 0|Z_i = 0, D^{obs}_i = 0]P[D^{obs}_i = 0|Z_i = 0]}{\pi_c} - \frac{\pi_n}{\pi_c} P[X_i = 0|Z_i = 0, D^{obs}_i = 0] =$$

$$\frac{P[Y^{obs}_{i1} = 1, X_i = 0|Z_i = 0, D^{obs}_i = 0](\pi_c + \pi_n)}{\pi_c} - \frac{\pi_n}{\pi_c} P[X_i = 0|Z_i = 0, D^{obs}_i = 0].$$

Analogous equivalence results can be derived for $U_{P^{(1\cdot)}_{c0}}$, as well as for upper and lower bounds for $P^{(1\cdot)}_{c1}$, $P^{(1\cdot)}_{n0}$, and $P^{(1\cdot)}_{a1}$.

Deriving bounds by conditioning on Y_2. Given the equivalence result obtained above, one may argue that a conditional strategy could be used also when the auxiliary variable is a secondary outcome. Here we show why a simple stratification on the observed value of Y_2 does not yield valid results. To see this, write $P^{(1\cdot)}_{c0}$ as

$$P^{(1\cdot)}_{c0} = P[Y_{i1}(0)|G_i = c, Y_{i2}(0) = 0] \cdot P[Y_{i2}(0) = 0|G_i = c] + P[Y_{i1}(0)|G_i = c, Y_{i2}(0) = 1] \cdot P[Y_{i2}(0) = 1|G_i = c]. \tag{52}$$

In order to bound $P[Y_{i1}(0)|G_i = c, Y_{i2}(0) = 0]$ and $P[Y_{i1}(0)|G_i = c, Y_{i2}(0) = 1]$ we need the strata proportions conditional on $Y_{i2}(0)$. These cannot be simply identified (as it is instead the case with a covariate) from Assumption 1. We can in fact identify the following conditional strata proportions:

$$P[G_i = a|Y_{i2}(0) = 1] = P[D^{obs}_i = 1|Z_i = 0, Y^{obs}_{i2} = 1],$$

and

$$P[G_i = n|Y_{i2}(1) = 1] = P[D^{obs}_i = 0|Z_i = 1, Y^{obs}_{i2} = 1],$$

which are in general different from $P[G_i = a|Y_{i2}(1) = 1]$ and $P[G_i = n|Y_{i2}(0) = 1]$, respectively. The proportion of compliers also differs depending on whether it is condition on $Y_{i2}(0) = 1$ or $Y_{i2}(1) = 1$; the conditional proportions can be identified by exploiting the additional exclusion restriction assumption. For example, $P[G_i = c|Y_{i2}(0) = 1]$ is identified as:

$$P[G_i = c|Y_{i2}(0) = 1] = P[G_i = c|Y_{i2}(0) = 1, Z_i = 0] = \frac{P[G_i = c|Z_i = 0]P(Y_{i2}(0) = 1|G_i = c, Z_i = 0)}{P(Y_{i2}(0) = 1|Z_i = 0)},$$

where $P(Y_{i2}(0) = 1|G_i = c, Z_i = 0)$ is identified (see Proposition 2) thank to the exclusion restriction.

This result shows that it would be feasible to derive bounds on quantities that are conditional on the values of the secondary outcome. However, this would not be as straightforward as with a covariate; we cannot simply stratify on the observed values of Y^{obs}_{i2}, but analysis must be conducted separately by treatment arm, and so conditional also on Z_i. In addition, while it is rather obvious that partial identification result requires the additional exclusion restriction for the secondary outcome, it is not so clear how the partial exclusion restriction helps tightening the bounds on ITT effects for the primary outcome. This contribution appears to be rather clear when working with joint probabilities for Y_{i1} and Y_{i2}. The ER imposes inequality constraints (as in (18) and (19)) on these joint probabilities that allow to remove values from the identified set that imply marginal probabilities that are not admissible.

Tables

Table 1: Perfect prediction of compliance status: bounds collapse

Underlying true marginal distribution

	Compliers		Never-Takers	
	Y_1	Y_2	Y_1	Y_2
π	0.6		0.4	
$Z = 0$	0.6	1	0.4	0
$Z = 1$	0.4	1	0.2	0
True ITT	**-0.2**	0	**-0.2**	0

Implied observed distribution

Z	D	Y_1	Y_2
0	0	0.52	0.6
1	0	0.2	0
1	1	0.4	1

Underlying joint true distributions under $Z = 0$

Compliers ($\rho = 0$)

	Y_2		
Y_1	1	0	
1	0.6	0	0.6
0	0.4	0	0.4
	1	0	1

Never-Takers ($\rho = 0$)

	Y_2		
Y_1	1	0	
1	0	0.4	0.4
0	0	0.6	0.6
	0	1	1

Implied observed distribution under $Z = 0$

	Y_2		
Y_1	1	0	
1	0.36	0.16	0.52
0	0.24	0.24	0.48
	0.6	0.6	1

ITT_c estimated under ER	-0.33
Bounds on ITT$_c$ without secondary outcome	$(-0.46; 0.2)$
Bounds on ITT$_n$ without secondary outcome	$(-0.8; 0.2)$
Bounds on ITT$_c$ with secondary outcome	$(\mathbf{-0.2}; \mathbf{-0.2})$
Bounds on ITT$_n$ with secondary outcome	$(\mathbf{-0.2}; \mathbf{-0.2})$

Table 2: Association with the primary outcome

Underlying true marginal distribution

	Compliers Y_1	Compliers Y_2	Never-Takers Y_1	Never-Takers Y_2
π	0.6		0.4	
$Z = 0$	0.7	0.7	0.5	0.5
$Z = 1$	0.4	0.4	0.4	0.5
True ITT	**-0.3**	−0.3	**-0.1**	0

Implied observed distribution

Z	D	Y_1	Y_2
0	0	0.62	0.62
1	0	0.4	0.5
1	1	0.4	0.4

(a) Strong dependence: bounds identify sign of ITTc and ITTn

Underlying joint true distributions under $Z = 0$

Compliers ($\rho = 0.8$)

Y_1	Y_2 = 1	Y_2 = 0	
1	0.65	0.05	0.7
0	0.05	0.25	0.3
	0.7	0.3	1

Never-Takers ($\rho = 0.9$)

Y_1	Y_2 = 1	Y_2 = 0	
1	0.48	0.02	0.5
0	0.02	0.48	0.5
	0.5	0.5	1

Implied observed distribution under $Z = 0$

Y_1	Y_2 = 1	Y_2 = 0	
1	0.58	0.04	0.62
0	0.04	0.34	0.38
	0.62	0.38	1

ITT_c estimated under ER	−0.37
Bounds on ITT_c without secondary outcome	$(-0.6; 0.03)$
Bounds on ITT_n without secondary outcome	$(-0.6; 0, 35)$
Bounds on ITT_c with secondary outcome	(**−0.36**; **−0.23**)
Bounds on ITT_n with secondary outcome	(**−0.19**; **−0.01**)

(b) Perfect dependence under Z=0: bounds collapse

Underlying joint true distributions under $Z = 0$

Compliers ($\rho = 1$)

Y_1	Y_2 = 1	Y_2 = 0	
1	0.7	0	0.7
0	0	0.3	0.3
	0.7	0.3	1

Never-Takers ($\rho = 1$)

Y_1	Y_2 = 1	Y_2 = 0	
1	0.5	0	0.5
0	0	0.5	0.5
	0.5	0.5	1

Implied observed distribution under $Z = 0$

Y_1	Y_2 = 1	Y_2 = 0	
1	0.62	0	0.62
0	0	0.38	0.38
	0.62	0.38	1

Bounds on ITT_c with secondary outcome	(**−0.3**; **−0.3**)
Bounds on ITT_n with secondary outcome	(**−0.1**; **−0.1**)

Table 3: Latent independence

Underlying true marginal distribution as in Table 2

	Compliers		Never-Takers	
	Y_1	Y_2	Y_1	Y_2
π	0.6		0.4	
$Z = 0$	0.7	0.7	0.5	0.5
$Z = 1$	0.4	0.4	0.4	0.5
True ITT	**-0.3**	−0.3	**-0.1**	0

Implied observed distribution

Z	D	Y_1	Y_2
0	0	0.62	0.62
1	0	0.4	0.5
1	1	0.4	0.4

Latent Independence: bounds do not collapse but ITT identified

Underlying joint true distributions under $Z = 0$

Compliers ($\rho = 0$)

	Y_2		
Y_1	1	0	
1	0.49	0.21	0.7
0	0.21	0.09	0.3
	0.7	0.3	1

Never-Takers ($\rho = 0$)

	Y_2		
Y_1	1	0	
1	0.25	0.25	0.5
0	0.25	0.25	0.5
	0.5	0.5	1

Implied observed distribution under $Z = 0$

	Y_2		
Y_1	1	0	
1	0.39	0.23	0.62
0	0.23	0.15	0.38
	0.62	0.38	1

ITT_c under LI assumptions	$\mathbf{-0.3}$
ITT_n under LI assumptions	$\mathbf{-0.1}$
Bounds on ITT_c with secondary outcome	$(\mathbf{-0.56}; \mathbf{0.03})$
Bounds on ITT_n with secondary outcome	$(\mathbf{-0.6}; \mathbf{0.29})$

Table 4: Influenza Vaccine encouragement study

Observed Marginal distributions

Z	D	Y_1	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
0	0	0.088	0.27	0.47	0.004	0.66	0.01	0.56	0.28	0.66
1	0	0.083	0.26	0.46	0.003	0.70	0.02	0.57	0.27	0.66
1	1	0.068	0.32	0.53	0.002	0.64	0.01	0.60	0.28	0.67
0	1	0.112	0.39	0.51	0.004	0.63	0.02	0.59	0.30	0.60
						$\rho_{Y_1 X_j}$				
1	1		0.04	−0.08	−0.01	−0.10	0.05	−0.31	0.03	0.01
0	0		0.05	−0.28	−0.02	−0.002	0.06	−0.33	0.04	0.01
π_c	0.69									
π_n	0.12									
π_a	0.19									

Y_1= flue-related hospital visits, X_1=COPD, X_2=age above median age, X_3=liver disease
X_4=sex, X_5=renal disease, X_6=heart disease, X_7=diabetes, X_8=race

ITT_c estimated under ER	-0.120
Bounds on ITT_c without auxiliary variables	$(-0.606; 0.176)$
Bounds on ITT_n without auxiliary variables	$(-0.020; 0.083)$
Bounds on ITT_a without auxiliary variables	$(-0.112; -0.002)$
$\bigcap$ of bounds on ITT_c with auxiliary covariates	$(\mathbf{-0.515}; \mathbf{0.176})$
$\bigcap$ of bounds on ITT_n with auxiliary covariates	$(\mathbf{-0.020}; \mathbf{0.068})$
$\bigcap$ of bounds on ITT_a with auxiliary covariates	$(\mathbf{-0.111}; \mathbf{-0.002})$

Table 5: Job Corps study

Observed Marginal distributions

Z	D	Y_1	Y_2
0	0	0.44	0.49
1	0	0.43	0.48
1	1	0.36	0.53
π_c	0.68		
π_n	0.32		
$\rho_{Y_1 Y_2}$	0.74		

Y_1= employment at week 52, Y_2= employment at week 130

Observed Joint Distribution under $Z = 0$

	Y_2		
Y_1	1	0	
1	0.40	0.04	0.44
0	0.09	0.47	0.56
	0.49	0.51	1

ITT_c estimated under ER	-0.09
Bounds on ITT_c without auxiliary variables	$(-0.29; 0.18)$
Bounds on ITT_n without auxiliary variables	$(-0.57; 0.43)$
Bounds on ITT_c with auxiliary outcome Y_2	$(\mathbf{-0.20}; \mathbf{-0.01})$
Bounds on ITT_n with secondary outcome Y_2	$(\mathbf{-0.18}; \mathbf{0.22})$

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