The relevant prefixes of coloured Motzkin walks: an average case analysis

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Abstract

In this paper we study some relevant prefixes of coloured Motzkin walks (otherwise called coloured Motzkin words). In these walks, the three kinds of step can have α, β and γ colours, respectively. In particular, when $\alpha = \beta = \gamma = 1$ we have the classical Motzkin walks while for $\alpha = \gamma = 1$ and $\beta = 0$ we find the well-known Dyck walks. By using the concept of Riordan arrays and probability generating functions we find the average length of the relevant prefix in a walk of length n and the corresponding variance in terms of α, β and γ. This result is interesting from a combinatorial point of view and also provides an average case analysis of algorithms related to the problem of ranking and generating uniformly at random the coloured Motzkin words.

1 Introduction

In this paper we consider the class of *coloured Motzkin walks*, that is, Motzkin walks in which each admissible step can have different colours. Classical Motzkin walks are well-known in combinatorics and derive their name from the Motzkin numbers, named after Theodore Motzkin [23]. The first few terms are $(M_n)_{n\in\mathbb{N}} = (1, 2, 4, 9, 21, 51, \cdots)$ and correspond to sequence A001006 in [29]. Many properties of these numbers can be found in [4]. Motzkin numbers enumerate various combinatorial objects and Donaghey and Shapiro [8] give 14 different manifestations of these numbers. In particular, they give the number of walks from $(0, 0)$ to $(n, 0)$ which never go below $y = 0$ and are made up only of the steps $(1, 1)$, $(1, 0)$ and $(1, -1)$. In the present paper, we consider a generalization of these Motzkin walks, where each step can be chosen between α, β and γ colours, respectively. When $\alpha = \beta = \gamma = 1$, we have the classical Motzkin walks while for $\alpha = \gamma = 1$ and $\beta = 0$ we find the well-known Dyck walks (see, e.g., [29, 31]). The use of colours for some of the steps in Motzkin walks (or words) can be found scattered in literature, mainly in the context of enumerative and bijective combinatorics (see, e.g., [3, 7, 20, 21, 25]).

In the last years, Motzkin walks (or other Motzkin objects) have attracted the attention of many people in the context of the random generation (see e.g., [9, 10, 11, 24]) and of the deterministic generation (see e.g., [16, 17, 26]) of combinatorial objects.

In particular, the random generation of the Motzkin left factors was studied in [5] and an algorithm for the random generation of Motzkin words was presented in [1]. Both papers use a rejection algorithm that behaves linearly on average. The general results in [9] give for this problem an approximate-size sampling of linear average complexity. The lexicographical generation of Motzkin words and the ranking and unranking of lexicographically ordered Motzkin words have been studied in [12, 14, 16, 17] among others (see also [13, 15]). Recently, a rejection algorithm for sampling at random a generalization of Motzkin walks according to their area has been studied in [6].

Some of the previous algorithms, in particular those presented in [12, 14], require to read the *minimal prefix* of a word, or its minimal suffix, in order to compute the position of the word according to the lexicographical order (ranking) or to determine the next word in the lexicographical order. For example, the strategy in [14] is to compute the number of lexicographically smaller words by reading the word from left to right until there is exactly one continuation to a word of the language. Therefore, the length of this prefix is important to analyze the performance of the algorithm. In particular, the average length of the prefix, provides an average case analysis of the algorithm. As we will see in this paper, this expression is also useful for the analysis of a simple algorithm for the uniform random generation of Motzkin walks which does not use a rejection technique. Our generation algorithm is based on a generating tree which encodes the Motzkin walks in such a way that the random generation of a walk coincides with the random generation of a particular walk in the tree. Quite obviously, this tree structure implies an order over the class of Motzkin walks and, in particular, if we read the paths in the generating tree from right to left (see Figure 2.1) we obtain exactly the lexicographical order described in [14]. The method behind this algorithm goes back to Nijenhuis and Wilf [24] and the explicit use of generating trees recalls the approach of [2], even if the generating tree under consideration in this paper does not exactly correspond to an ECO rule. In order to generate a walk of length n, the algorithm performs a number of calls to a function `random()` which generates uniformly a real number λ with $0 \leq \lambda < 1$. One interesting point is the fact that the average number of calls to `random()` performed by the algorithm coincides with the average length of the relevant prefixes of coloured Motzkin walks.

For the reasons exposed above, the aim of this paper is to give an accurate analysis of the average length of the relevant prefixes in coloured Motzkin walks as a function of α, β and γ. Our analysis is based on properties of the coloured Motzkin triangle, due to its *Riordan array* nature, and uses probability generating functions to compute the mean, the variance and, if one desires, higher moments. The method is linear, elegant and simple and allows to generalize and unify the analysis performed in [14] in the particular cases $\alpha = \beta = \gamma = 1$ and $\alpha = \gamma = 1$, $\beta = 0$.

Section 2 is devoted to the analysis of some properties of the Motzkin triangle. These properties are used in Section 3 to find the average length and the variance of the relevant prefixes. Finally, in Section 4 we give a general algorithm for the random generation of coloured Motzkin walks. Although the method behind the algorithm is already known in the literature, its application to coloured Motzkin walks, as far as we know, is new. Moreover, we show how the analysis of the algorithm is an immediate consequence of the results in Section 3.

2 The coloured Motzkin walks

In this paper we consider the following model of walks: each walk of length n is represented by a sequence $((0, y_0), (1, y_1), \cdots, (n, y_n))$ of $n+1$ points in $\mathbb{N}^2$, where $y_0 = 0$ and $y_k = y_{k-1} + s_k$ where, when $y_{k-1} = j$, the s_k are constrained to belong to a fixed set $\mathcal{S}_j$. In particular we have $\mathcal{S}_j = \{-1^\gamma, 0^\beta, 1^\alpha\}$ for $j > 0$ and $\mathcal{S}_0 = \{0^\beta, 1^\alpha\}$, where α, β and γ represent the number of colours for each kind of step ($\alpha, \gamma \neq 0$). We can represent all the walks of length $\leq n$ as a tree of height n, where the root (at level 0 by convention) is labelled with 0 and where the label of each node at level n encodes a possible position of the walk. More precisely, a walk of length n corresponds to a branch of length $n+1$ in the generating tree defined by the following rule:

$$\begin{cases} \mathit{root}: & (0) \\ \mathit{rule}: & (k) \quad \rightarrow (k-1)^\gamma (k)^\beta (k+1)^\alpha \end{cases} \tag{2.1}$$

Figure 2.1 illustrates the generating tree corresponding to (2.1) up to level $n = 4$. In particular, in the figure some nodes with the same label have been collected, so that a node with label j^p denotes p nodes with label j. Moreover, at level 4 of the tree, only the nodes with label 0 are explicitly represented.

Let $M^{[\alpha,\beta,\gamma]} = (M^{[\alpha,\beta,\gamma]}_{n,k})_{n,k \in N}$ be the matrix associated with the generating tree described by (2.1), that is, $M^{[\alpha,\beta,\gamma]}_{n,k}$ counts the number of nodes at level n having label k. This kind of matrices has been studied in [22] where the authors proved under which conditions the matrix associated to a given generating tree is a Riordan array and vice versa. The concept of a *Riordan array* has been introduced in [28] with the aim of defining a class of infinite lower triangular arrays with properties analogous to those of the Pascal triangle. This concept has been successively studied in [30] in the context of the computation of combinatorial sums. Some other aspects of the theory have been studied in [19] and the literature about Riordan arrays is vast and still growing.

Formally, a Riordan array is an infinite lower triangular array $(d_{n,k})_{n,k \in \mathbb{N}}$, defined by a pair of formal power series $(d(t), h(t))$, such that the generic element $d_{n,k}$ is the n-th coefficient in the series $d(t)(th(t))^k$, i.e.:

$$d_{n,k} = [t^n] d(t)(th(t))^k, \qquad n, k \geq 0.$$

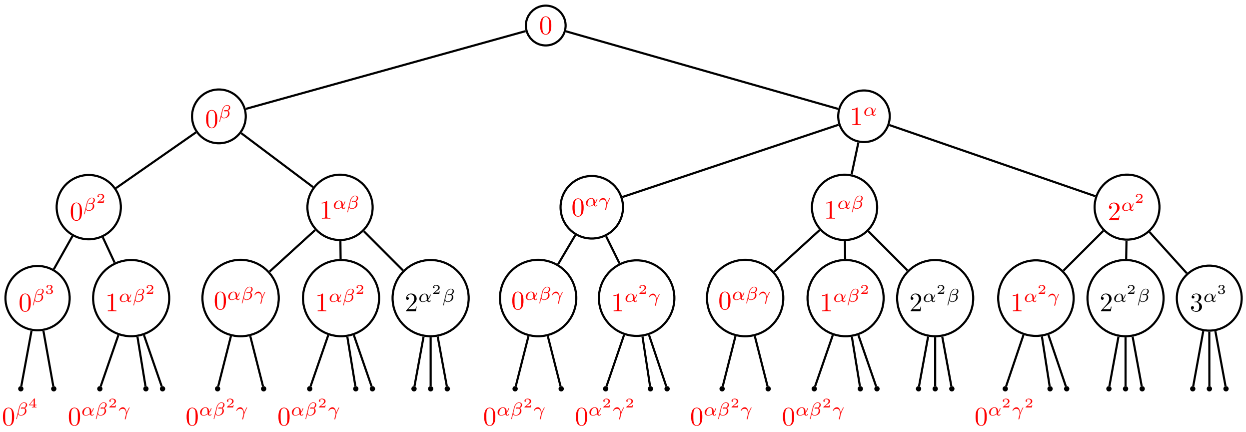

Figure 2.1: The generating tree corresponding to rule (2.1) up to level $n = 4$.

From this definition we have $d_{n,k} = 0$ for $k > n$ (an equivalent definition includes the t inside the function $h(t)$ and requires $h(0) = 0$). An alternative definition is in terms of the so-called A-sequence and Z-sequence, with generating functions $A(t)$ and $Z(t)$ satisfying the relations:

$$h(t) = A(th(t)), \quad d(t) = \frac{d_0}{1 - tZ(th(t))} \quad \text{with} \quad d_0 = d(0).$$

The general theory of Riordan arrays and the proofs of their properties can be found in [18, 19].

By using the results in [22] we can easily find that $M^{[\alpha,\beta,\gamma]}$ is the Riordan array defined by the pair $(d^{[\alpha,\beta,\gamma]}(t), h^{[\alpha,\beta,\gamma]}(t))$ such that:

$$d^{[\alpha,\beta,\gamma]}(t) = \frac{1 - \beta t - \sqrt{1 - 2\beta t + (\beta^2 - 4\alpha\gamma)t^2}}{2\alpha\gamma t^2}, \quad h^{[\alpha,\beta,\gamma]}(t) = \alpha d^{[\alpha,\beta,\gamma]}(t),$$

hence,

$$M^{[\alpha,\beta,\gamma]}_{n,k} = \alpha^k [t^{n-k}] \left(\frac{1 - \beta t - \sqrt{1 - 2\beta t + (\beta^2 - 4\alpha\gamma)t^2}}{2\alpha\gamma t^2} \right)^{k+1}. \tag{2.2}$$

The A and Z-sequences of the array are defined by the relations:

$$h^{[\alpha,\beta,\gamma]}(t) = A^{[\alpha,\beta,\gamma]}(th^{[\alpha,\beta,\gamma]}(t)), \quad d^{[\alpha,\beta,\gamma]}(t) = \frac{1}{1 - tZ^{[\alpha,\beta,\gamma]}(th^{[\alpha,\beta,\gamma]}(t))}$$

and correspond to the generating functions:

$$A^{[\alpha,\beta,\gamma]}(t) = \alpha + \beta t + \gamma t^2, \quad Z^{[\alpha,\beta,\gamma]}(t) = \beta + \gamma t$$

which translate into the recurrence relations:

$$M^{[\alpha,\beta,\gamma]}_{n+1,k+1} = \alpha M^{[\alpha,\beta,\gamma]}_{n,k} + \beta M^{[\alpha,\beta,\gamma]}_{n,k+1} + \gamma M^{[\alpha,\beta,\gamma]}_{n,k+2}, \quad M^{[\alpha,\beta,\gamma]}_{n+1,0} = \beta M^{[\alpha,\beta,\gamma]}_{n,0} + \gamma M^{[\alpha,\beta,\gamma]}_{n,1} \tag{2.3}$$

with initial condition $M^{[\alpha,\beta,\gamma]}_{0,0} = 1$. The first 6 rows of matrix $M^{[\alpha,\beta,\gamma]}$ are shown in Table 2.1.

Since a path of length $n+1$ in the tree (2.1) encodes a possible walk of length n in our model, we have that $M^{[\alpha,\beta,\gamma]}_{n,k}$ also counts the number of walks of length n ending at altitude k. In particular, $M^{[\alpha,\beta,\gamma]}_{n,0}$ counts the number of coloured walks of length n coming back to 0 and we have:

$$M^{[\alpha,\beta,\gamma]}_{n,0} = M^{[\alpha,\beta,\gamma]}_{n} = [t^n] d^{[\alpha,\beta,\gamma]}(t) = [t^n] \frac{1 - \beta t - \sqrt{1 - 2\beta t + (\beta^2 - 4\alpha\gamma)t^2}}{2\alpha\gamma t^2}.$$

In particular, numbers $M^{[1,1,1]}_n$ are well-known and called *Motzkin numbers* in the literature; matrix $M^{[1,1,1]}$ is known as the *Motzkin triangle* and is illustrated in Table 2.2 (see, e.g., [4]). Another well-known triangle is $M^{[1,0,1]}$ which is illustrated in Table 2.3 and corresponds to the aerated *Catalan triangle*

n/k	0	1	2	3	4	5
0	1					
1	β	α				
2	$\beta^2+\alpha\gamma$	$2\alpha\beta$	α^2			
3	$\beta(\beta^2+3\alpha\gamma)$	$\alpha(2\alpha\gamma+3\beta^2)$	$3\alpha^2\beta$	α^3		
4	$\beta^4+6\alpha\beta^2\gamma+2\alpha^2\gamma^2$	$4\alpha\beta(\beta^2+2\alpha\gamma)$	$3\alpha^2(\alpha\gamma+2\beta^2)$	$4\alpha^3\beta$	α^4	
5	$\beta(\beta^4+10\alpha\beta^2\gamma+10\alpha^2\gamma^2)$	$5\alpha(\beta^4+4\alpha\beta^2\gamma+\alpha^2\gamma^2)$	$5\alpha^2\beta(2\beta^2+3\alpha\gamma)$	$2\alpha^3(2\alpha\gamma+5\beta^2)$	$5\alpha^4\beta$	α^5

Table 2.1: The triangle $M^{[\alpha,\beta,\gamma]}$.

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	2	2	1					
3	4	5	3	1				
4	9	12	9	4	1			
5	21	30	25	14	5	1		
6	51	76	69	44	20	6	1	
7	127	196	189	133	70	27	7	1

Table 2.2: The triangle $M^{[1,1,1]}$

(see, e.g., [29]). Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ appear in the first column of the triangle, that is, $M_n^{[1,0,1]}$ is equal to $C_{n/2}$ if n is even, and is zero otherwise.

We now give some results which will be useful in the next sections.

Theorem 2.1 *We have the following formulae for coloured Motzkin numbers:*

$$M_n^{[\alpha,\beta,\gamma]} \sim \frac{(\beta+2\sqrt{\alpha\gamma})^{(n+3/2)}}{(\alpha\gamma)^{3/4}(2n-1)4^n}\binom{2n}{n}\left(1-3\frac{\beta+14\sqrt{\alpha\gamma}}{8\sqrt{\alpha\gamma}(2n-3)}\right) \quad \textit{if} \quad \beta \neq 0,$$

$$M_{2m}^{[\alpha,0,\gamma]} = \frac{(\alpha\gamma)^m}{m+1}\binom{2m}{m}, \quad M_{2m+1}^{[\alpha,0,\gamma]} = 0 \quad \textit{if} \quad \beta = 0.$$

Proof: The dominant singularity $s^{[\alpha,\beta,\gamma]}$ of $d^{[\alpha,\beta,\gamma]}(t)$ is the minimal root, in modulus, of the equation $1-2\beta t+(\beta^2-4\alpha\gamma)t^2$. The roots are $1/(\beta+2\sqrt{\alpha\gamma})$ and $1/(\beta-2\sqrt{\alpha\gamma})$, so, when $\beta \neq 0$, the singularity is $s^{[\alpha,\beta,\gamma]} = 1/(\beta+2\sqrt{\alpha\gamma})$. The asymptotic approximation of $M_n^{[\alpha,\beta,\gamma]}$ can be found by performing a series development of $d^{[\alpha,\beta,\gamma]}(t)$ around it. If we put $t=(1-w)s^{[\alpha,\beta,\gamma]}$ in $d^{[\alpha,\beta,\gamma]}(t)$, so that $w = 1-t/s^{[\alpha,\beta,\gamma]}$, and then compute the series development around $w=0$, we get (omitting the constant and the w^k terms,

n/k	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	1	0	1					
3	0	2	0	1				
4	2	0	3	0	1			
5	0	5	0	4	0	1		
6	5	0	9	0	5	0	1	
7	0	14	0	14	0	6	0	1

Table 2.3: The triangle $M^{[1,0,1]}$

$k \in \mathbb{N}$, which have no asymptotic relevance):

$$d^{[\alpha,\beta,\gamma]}(t) = -\frac{(\beta+2\sqrt{\alpha\gamma})^{3/2}}{(\alpha\gamma)^{3/4}}\sqrt{w} - \frac{(\beta+2\sqrt{\alpha\gamma})^{3/2}(\beta+14\sqrt{\alpha\gamma})}{8(\alpha\gamma)^{5/4}}w^{3/2} + O(w^2)$$

Hence we have:

$$d^{[\alpha,\beta,\gamma]}(t) \sim -\frac{(\beta+2\sqrt{\alpha\gamma})^{3/2}}{(\alpha\gamma)^{3/4}}\sqrt{1-\frac{t}{s^{[\alpha,\beta,\gamma]}}} - \frac{(\beta+2\sqrt{\alpha\gamma})^{3/2}(\beta+14\sqrt{\alpha\gamma})}{8(\alpha\gamma)^{5/4}}\left(1-\frac{t}{s^{[\alpha,\beta,\gamma]}}\right)^{3/2}$$

and therefore:

$$\begin{aligned} M_n^{[\alpha,\beta,\gamma]} \sim & -\frac{(\beta+2\sqrt{\alpha\gamma})^{3/2}}{(\alpha\gamma)^{3/4}}\binom{1/2}{n}\left(\frac{-1}{s^{[\alpha,\beta,\gamma]}}\right)^n + \\ & -\frac{(\beta+2\sqrt{\alpha\gamma})^{3/2}(\beta+14\sqrt{\alpha\gamma})}{8(\alpha\gamma)^{5/4}}\binom{3/2}{n}\left(\frac{-1}{s^{[\alpha,\beta,\gamma]}}\right)^n. \end{aligned} \tag{2.4}$$

After some simplification, we obtain the formula in the statement of the theorem. On the other hand, when $\beta = 0$, we have:

$$d^{[\alpha,0,\gamma]}(t) = \frac{1-\sqrt{1-4\alpha\gamma t^2}}{2\alpha\gamma t^2}$$

and, therefore, for $n = 2m$ we obtain:

$$M_{2m}^{[\alpha,0,\gamma]} = [t^{2m}]\frac{1-\sqrt{1-4\alpha\gamma t^2}}{2\alpha\gamma t^2} = [t^m]\frac{1-\sqrt{1-4\alpha\gamma t}}{2\alpha\gamma t} = \frac{(\alpha\gamma)^m}{m+1}\binom{2m}{m}.$$

■

The next result concerns the column generating functions of the coloured Motzkin triangle $M^{[\alpha,\beta,\gamma]}$:

Theorem 2.2 *Let $f_j(t)$ be the generating function of column j in the coloured Motzkin triangle. Then we have:*

$$f_0(t) = d^{[\alpha,\beta,\gamma]}(t), \quad f_j(t) = \frac{p_j(t)d^{[\alpha,\beta,\gamma]}(t) - p_{j-1}(t)}{\alpha^j t^j} \quad \text{for} \quad j \geq 1$$

where the $p_j(t)$ are polynomials defined as follows:

$$p_j(t) = [w^j]F(t,w), \quad F(t,w) = \frac{1}{1-(1-\beta t)w + \alpha\gamma t^2 w^2}.$$

Proof: By a first application of rule (2.1) with $k = 0$ we have $(0) \to (0)^\beta(1)^\alpha$ and, consequently,

$$f_0(t) = 1 + t(\beta f_0(t) + \alpha f_1(t)), \quad \text{or} \quad f_1(t) = \frac{(1-\beta t)f_0(t) - 1}{\alpha t}.$$

From $(1) \to (0)^\gamma(1)^\beta(2)^\alpha$ we deduce:

$$f_1(t) = t(\gamma f_0(t) + \beta f_1(t) + \alpha f_2(t)), \quad \text{or} \quad f_2(t) = \frac{(1-2\beta t+(\beta^2-\alpha\gamma)t^2)f_0(t) - (1-\beta t)}{\alpha^2 t^2}.$$

In general, for $j \geq 1$, we have:

$$f_j(t) = t(\gamma f_{j-1}(t) + \beta f_j(t) + \alpha f_{j+1}(t)) \quad \text{or} \quad f_{j+1}(t) = \frac{(1-\beta t)f_j(t) - \gamma t f_{j-1}(t)}{\alpha t}$$

and by repeated substitutions we find:

$$f_j(t) = \frac{p_j(t)f_0(t) - p_{j-1}(t)}{\alpha^j t^j},$$

where $p_j(t)$ is a polynomial of degree j. Therefore we have:

$$\frac{p_{j+1}(t)f_0(t)-p_j(t)}{\alpha^{j+1}t^{j+1}}=\frac{((1-\beta t)p_j(t)-\alpha\gamma t^2p_{j-1}(t))f_0(t)-((1-\beta t)p_{j-1}(t)-\alpha\gamma t^2p_{j-2}(t))}{\alpha^{j+1}t^{j+1}},$$

and, equivalently:

$$p_{j+1}(t)=(1-\beta t)p_j(t)-\alpha\gamma t^2p_{j-1}(t),\quad p_0(t)=1,\quad p_1(t)=1-\beta t.$$

Therefore, by setting $F(t,w)=\sum_{j\geq 0}p_j(t)w^j$, we have:

$$\frac{F(t,w)-p_0(t)}{w}=(1-\beta t)F(t,w)-\alpha\gamma t^2wF(t,w),\quad \text{or}\quad F(t,w)=\frac{1}{1-(1-\beta t)w+\alpha\gamma t^2w^2}.$$

■

The previous theorem allows us to compute the numbers $M_{n,k}^{[\alpha,\beta,\gamma]}$ as a linear combination of the elements in the first column of the triangle, that is, of coloured Motzkin numbers $M_n^{[\alpha,\beta,\gamma]}$:

Corollary 2.3 *The elements of the coloured Motzkin triangle satisfy the following formula:*

$$M_{n,k}^{[\alpha,\beta,\gamma]}=\frac{1}{\alpha^k}[t^{n+k}]p_k(t)d^{[\alpha,\beta,\gamma]}(t),\quad p_k(t)=[w^k]\frac{1}{1-(1-\beta t)w+\alpha\gamma t^2w^2}.$$

In this paper we are interested in the study of what we call a *relevant prefix* of a coloured Motzkin word; therefore, before concluding this section, we give a formal definition of this quantity.

Definition 2.4 (Relevant prefix) *The relevant prefix of a coloured Motzkin walk is the minimal subwalk such that there is exactly one continuation to a coloured Motzkin walk.*

In other words, the steps that follow the relevant prefix are uniquely determined by the length of the walk. In Figure 2.2 we illustrate the Motzkin walks of length 4 and their relevant prefixes, in the case $\alpha=\beta=\gamma=1$.

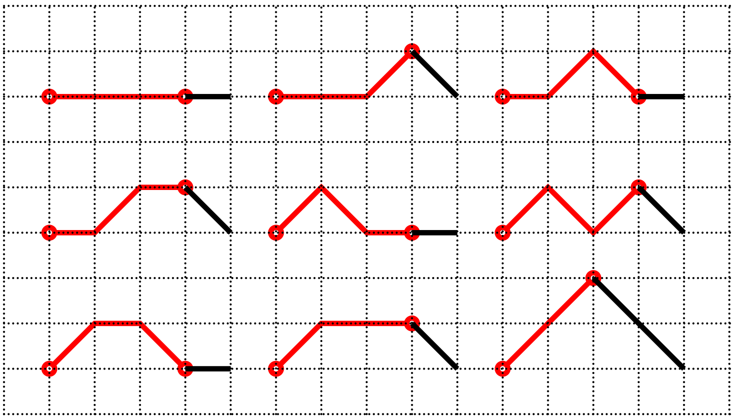

Figure 2.2: Motzkin walks of lenght $n=4$ and their relevant prefixes.

3 The average length of the relevant prefixes

In this section, we concentrate on the study of the average length of the relevant prefixes of coloured Motzkin walks and the corresponding variance. The main results are given by formulae (3.2), (3.4), (3.5) and (3.6) and Theorems 3.9 and 3.10.

Our analysis is based on the concept of *probability generating functions* which provides a way to find the average value and the variance of a given probability distribution in an elegant and quite simple way (see, e.g., [27]). For the sake of simplicity, we recall that given a random variable X that takes on only nonnegative values, with $p_k \equiv Pr\{X = k\}$, the function $P(u) = \sum_{k\geq 0} p_k u^k$ is called the probability generating function for the random variable. The mean value and the variance of X can be found by computing $P'(1)$ and $P''(1) + P'(1) - P'(1)^2$, respectively. In particular, in the present paper, we study the probability generating function

$$P_n^{[\alpha,\beta,\gamma]}(u) = \pi_0^{[\alpha,\beta,\gamma]} + \pi_1^{[\alpha,\beta,\gamma]} u + \cdots + \pi_{n-2}^{[\alpha,\beta,\gamma]} u^{n-2} + \pi_{n-1}^{[\alpha,\beta,\gamma]} u^{n-1}.$$

where $\pi_i^{[\alpha,\beta,\gamma]}$ denotes the probability that the relevant prefix of a coloured Motzkin walk of length n has length i.

By using a combinatorial reasoning, we have:

Theorem 3.1 *Let $\pi_i^{[\alpha,\beta,\gamma]}$ be the probability that the relevant prefix of a coloured Motzkin walk of length n has length i. We have, for $\beta \neq 0$ and $i < n-1$:*

$$\pi_i^{[\alpha,\beta,\gamma]} = \frac{\gamma^{n-i} M_{i,n-i}^{[\alpha,\beta,\gamma]} - \gamma^{n-i+1} M_{i-1,n-i+1}^{[\alpha,\beta,\gamma]}}{M_n^{[\alpha,\beta,\gamma]}},$$

and, for $\beta \neq 0$ and $i = n-1$:

$$\pi_{n-1}^{[\alpha,\beta,\gamma]} = 1 - \sum_{i=0}^{n-2} \pi_i^{[\alpha,\beta,\gamma]}.$$

Proof: The fact that a relevant prefix of a coloured Motzkin walk has length $i \leq n-2$ corresponds to a walk in the generating tree which starts from the root, stops at level i with a node with label $n-i$, and finally is followed by a sequence of $n-i$ nodes with decreasing labels. In fact, the label of a node corresponds to the altitude of the coloured Motzkin walk after i steps and the walk must end at altitude 0. At level i there are $M_{i,n-i}^{[\alpha,\beta,\gamma]}$ nodes with label $n-i$, so the probability for a node to have label $n-i$ can be found by dividing that value by $M_n^{[\alpha,\beta,\gamma]}$. Moreover, we have to subtract the probability to generate a label $n-(i-1)$ at level $i-1$, because this value would arrest the process. Each of the $n-j$ nodes with decreasing labels can be chosen among γ colours and this explains the γ^{n-j} factors. Finally, the last node has always label 0 and therefore the last step of the walk is an horizontal or a down step according to the value of the label in the last but one node. Therefore, the length of the relevant prefix is at most $n-1$. ■

When $\beta = 0$, we can do better since in this case the length of the relevant prefix is at most $n-2$. In fact, the last two steps of a coloured Dyck walk are both down steps or correspond to an up and down sequence of steps, according to the label of the node at level $n-2$. Therefore, we have:

Corollary 3.2 *Let $\pi_i^{[\alpha,0,\gamma]}$ be the probability that the relevant prefix of a coloured Dyck walk of length n has length i. We have, for $i < n-2$:*

$$\pi_i^{[\alpha,0,\gamma]} = \frac{\gamma^{n-i} M_{i,n-i}^{[\alpha,0,\gamma]} - \gamma^{n-i+1} M_{i-1,n-i+1}^{[\alpha,0,\gamma]}}{M_n^{[\alpha,0,\gamma]}},$$

and, for $i = n-1$:

$$\pi_{n-2}^{[\alpha,0,\gamma]} = 1 - \sum_{i=0}^{n-3} \pi_i^{[\alpha,0,\gamma]}.$$

In the rest of the paper, for the sake of notation, we consider $\beta \neq 0$ if the value β is not explicitly denoted with zero. In other words, the superscript $[\alpha,\beta,\gamma]$ concerns coloured Motzkin walks with $\beta \neq 0$ while $[\alpha,0,\gamma]$ concerns coloured Dyck walks. We will also use $[\alpha,\beta=0,\gamma]$ to denote an expression corresponding to coloured Motzkin walks evaluated in $\beta = 0$.

As a consequence of Theorem 3.1 and Corollary 3.2, we have:

Corollary 3.3 *Let $\pi_i^{[\alpha,0,\gamma]}$ be the probability that the relevant prefix of a coloured Dyck walk of length n has length i, for $i \leq n-2$, and let $P_n^{[\alpha,0,\gamma]}(u)$ be the corresponding probability generating function. We have:*

$$P_n^{[\alpha,0,\gamma]}(u) = P_n^{[\alpha,\beta=0,\gamma]}(u) + \pi_{n-1}^{[\alpha,\beta=0,\gamma]} u^{n-2} - \pi_{n-1}^{[\alpha,\beta=0,\gamma]} u^{n-1}$$

where $P_n^{[\alpha,\beta=0,\gamma]}(u)$ and $\pi_{n-1}^{[\alpha,\beta=0,\gamma]}$ correspond to Theorem 3.1 and must be valuated in $\beta = 0$.

Therefore, we can compute $P_n^{[\alpha,\beta,\gamma]}(u)$ according to Theorem 3.1 and then use Corollary (3.3) to study the case $\beta = 0$. In order to evaluate the average length of the relevant prefixes we need to differentiate $P_n^{[\alpha,\beta,\gamma]}(u)$ and compute $P_n^{[\alpha,\beta,\gamma]'}(1)$. However, it is simpler to work with

$$\mathcal{P}_n^{[\alpha,\beta,\gamma]}(u) = M_n^{[\alpha,\beta,\gamma]} P_n^{[\alpha,\beta,\gamma]}(u) = \sum_{i \geq 0} \Pi_i^{[\alpha,\beta,\gamma]} u^i.$$

In other words, $\Pi_i^{[\alpha,\beta,\gamma]}$ denotes the frequency of coloured Motzkin walks of length n with relevant prefix of length i. We have the following result:

Theorem 3.4 *Let $\Pi_i^{[\alpha,\beta,\gamma]} = M_n^{[\alpha,\beta,\gamma]} \pi_i^{[\alpha,\beta,\gamma]}$; then we have:*

$$\Pi_i^{[\alpha,\beta,\gamma]} = \frac{1}{\alpha\gamma}[t^{n+2}](\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^{n-i+1}(1 - \alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t)) \quad for \quad i = 0, \cdots, n-2.$$

Moreover,

$$\Pi_{n-1}^{[\alpha,\beta,\gamma]} = M_n^{[\alpha,\beta,\gamma]} - \frac{1}{\alpha\gamma}[t^{n+2}](\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^3 = M_n^{[\alpha,\beta,\gamma]} - \gamma^2 M_{n-2,2}^{[\alpha,\beta,\gamma]}.$$

Proof: From Theorem 3.1 and formula (2.2), we have:

$$\Pi_i^{[\alpha,\beta,\gamma]} = \gamma^{n-i} M_{i,n-i}^{[\alpha,\beta,\gamma]} - \gamma^{n-i+1} M_{i-1,n-i+1}^{[\alpha,\beta,\gamma]} =$$

$$= (\alpha\gamma)^{n-i}[t^{i-(n-i)}]d^{[\alpha,\beta,\gamma]}(t)^{n-i+1} - (\alpha\gamma)^{n-i+1}[t^{i-1-(n-i+1)}]d^{[\alpha,\beta,\gamma]}(t)^{n-i+2} =$$

$$= (\alpha\gamma)^{n-i}[t^{2i-n}]\frac{(t^2 d^{[\alpha,\beta,\gamma]}(t))^{n-i+1}}{t^{2(n-i+1)}} - (\alpha\gamma)^{n-i+1}[t^{2i-2-n}]\frac{(t^2 d^{[\alpha,\beta,\gamma]}(t))^{n-i+2}}{t^{2(n-i+2)}},$$

hence for $i < n-1$ ::

$$\Pi_i{}^{[\alpha,\beta,\gamma]} = (\alpha\gamma)^{n-i}[t^{n+2}]\left(t^2 d^{[\alpha,\beta,\gamma]}(t)\right)^{n-i+1} - (\alpha\gamma)^{n-i+1}[t^{n+2}]\left(t^2 d^{[\alpha,\beta,\gamma]}(t)\right)^{n-i+2}.$$

In order to compute $\Pi_n^{[\alpha,\beta,\gamma]} = M_n^{[\alpha,\beta,\gamma]} - \sum_{i=0}^{n-2} \Pi_i{}^{[\alpha,\beta,\gamma]}$ we first evaluate the following sum:

$$\sum_{i=0}^{n-2} \Pi_i{}^{[\alpha,\beta,\gamma]} = \frac{1}{\alpha\gamma}[t^{n+2}](1 - \alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))(\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^{n+1} \sum_{i=0}^{n-2}(\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^{-i}.$$

The sum in the right hand side can be easily evaluated, since it is the sum of a geometric progression, and after some simplification we have:

$$\sum_{i=0}^{n-2} \Pi_i{}^{[\alpha,\beta,\gamma]} = \frac{1}{\alpha\gamma}[t^{n+2}]\left((\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^3 - (\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^{n+2}\right).$$

The second term within parentheses is equal to zero since the corresponding generating function has order greater than $n+2$ and finally by (2.2) we have:

$$\Pi_{n-1}^{[\alpha,\beta,\gamma]} = M_n^{[\alpha,\beta,\gamma]} - \frac{1}{\alpha\gamma}[t^{n+2}](\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^3 =$$

$$= M_n^{[\alpha,\beta,\gamma]} - (\alpha\gamma)^2[t^{n-4}](d^{[\alpha,\beta,\gamma]}(t))^3 = M_n^{[\alpha,\beta,\gamma]} - \gamma^2 M_{n-2,2}^{[\alpha,\beta,\gamma]}.$$

■

We have computed $\Pi_{n-1}^{[\alpha,\beta,\gamma]}$ by subtraction. However, by using formulae (2.3) we note:

$$M_{n,0}^{[\alpha,\beta,\gamma]} = \beta M_{n-1,0}^{[\alpha,\beta,\gamma]} + \gamma M_{n-1,1}^{[\alpha,\beta,\gamma]} =$$

$$= \beta \left(\beta M_{n-2,0}^{[\alpha,\beta,\gamma]} + \gamma M_{n-2,1}^{[\alpha,\beta,\gamma]} \right) + \gamma \left(\alpha M_{n-2,0}^{[\alpha,\beta,\gamma]} + \beta M_{n-2,1}^{[\alpha,\beta,\gamma]} + \gamma M_{n-2,2}^{[\alpha,\beta,\gamma]} \right) =$$

$$= (\beta^2 + \alpha\gamma) M_{n-2,0}^{[\alpha,\beta,\gamma]} + 2\beta\gamma M_{n-2,1}^{[\alpha,\beta,\gamma]} + \gamma^2 M_{n-2,2}^{[\alpha,\beta,\gamma]},$$

therefore:

$$\Pi_{n-1}^{[\alpha,\beta,\gamma]} = M_n^{[\alpha,\beta,\gamma]} - \gamma^2 M_{n-2,2}^{[\alpha,\beta,\gamma]} = (\beta^2 + \alpha\gamma) M_{n-2,0}^{[\alpha,\beta,\gamma]} + 2\beta\gamma M_{n-2,1}^{[\alpha,\beta,\gamma]}. \tag{3.1}$$

This formula has a direct combinatorial interpretation. In fact, the coloured Motzkin walks which have a relevant prefix of length $n-1$ correspond to the walks which after $n-2$ steps arrive i) at altitude 0 and then are followed by two horizontal steps or by an up step followed by a down step; ii) at altitude 1 and then are followed by an horizonal step and a down step, or vice versa.

By using the results of Theorem 3.4 and after some simplification, we obtain the following formula for $\mathcal{P}_n^{[\alpha,\beta,\gamma]}(u)$:

Theorem 3.5 *Let* $\mathcal{P}_n^{[\alpha,\beta,\gamma]}(u) = M_n^{[\alpha,\beta,\gamma]} P_n^{[\alpha,\beta,\gamma]}(u)$; *then we have:*

$$\mathcal{P}_n^{[\alpha,\beta,\gamma]}(u) = M_n^{[\alpha,\beta,\gamma]} u^{n-1} - \frac{1}{\alpha\gamma}(u-1)u^{n-1}[t^{n+2}] \frac{(\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^3}{u - \alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t)}$$

We are now able to compute the average frequency $\mathcal{P}_n^{[\alpha,\beta,\gamma]\prime}(1)$:

Theorem 3.6 *Let* $\mathcal{P}_n^{[\alpha,\beta,\gamma]}(u) = M_n^{[\alpha,\beta,\gamma]} P_n^{[\alpha,\beta,\gamma]}(u)$; *then we have:*

$$\mathcal{P}_n^{[\alpha,\beta,\gamma]\prime}(1) = (n-1) M_n^{[\alpha,\beta,\gamma]} - [t^{n+2}] f^{[\alpha,\beta,\gamma]}(t)$$

where

$$f^{[\alpha,\beta,\gamma]}(t) = \frac{(\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^3}{\alpha\gamma(1 - \alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))}.$$

Proof: By differentiating $\mathcal{P}_n^{[\alpha,\beta,\gamma]}(u)$ with respect to u we have:

$$\mathcal{P}_n^{[\alpha,\beta,\gamma]\prime}(u) = (n-1) M_n^{[\alpha,\beta,\gamma]} u^{n-2} - \frac{u^{n-1}}{\alpha\gamma}[t^{n+2}] \frac{(\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^3}{u - \alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t)} +$$

$$-\frac{(n-1)(u-1)u^{n-2}}{\alpha\gamma}[t^{n+2}] \frac{(\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^3}{u - \alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t)} + \frac{(u-1)u^{n-1}}{\alpha\gamma}[t^{n+2}] \frac{(\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^3}{(u - \alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^2}$$

and then setting $u = 1$, we find:

$$\mathcal{P}_n^{[\alpha,\beta,\gamma]\prime}(1) = (n-1) M_n^{[\alpha,\beta,\gamma]} - [t^{n+2}] \frac{(\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^3}{\alpha\gamma(1 - \alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))}.$$

■

From the previous theorem we have that the average length $\mu_n^{[\alpha,\beta,\gamma]}$ of the relevant prefixes in coloured Motzkin words of length n is given by

$$\mu_n^{[\alpha,\beta,\gamma]} = P_n^{[\alpha,\beta,\gamma]\prime}(1) = n - 1 - [t^{n+2}] f^{[\alpha,\beta,\gamma]}(t) / M_n^{[\alpha,\beta,\gamma]} \tag{3.2}$$

When $\beta = 0$, by using Corollary 3.3 and formula (3.1), we have

$$P_n^{[\alpha,0,\gamma]'}(u) = P_n^{[\alpha,\beta=0,\gamma]'}(u) - \frac{\alpha\gamma M_{n-2,0}^{[\alpha,0,\gamma]}}{M_n^{[\alpha,0,\gamma]}}\left((n-2)u^{n-3} - (n-1)u^{n-2}\right) \tag{3.3}$$

and consequently the average length $\mu_n^{[\alpha,0,\gamma]}$ of the relevant prefixes in coloured Dyck words of length $n = 2m$ is given by:

$$\mu_{2m}^{[\alpha,0,\gamma]} = \mu_{2m}^{[\alpha,\beta=0,\gamma]} - \frac{m+1}{2(2m-1)} \tag{3.4}$$

Now, in order to compute the variance, we need to differentiate twice $P_n^{[\alpha,\beta,\gamma]}(u)$ and compute $\mathcal{P}_n^{[\alpha,\beta,\gamma]''}(1)$; after some computations we have:

Theorem 3.7 *Let $\mathcal{P}_n^{[\alpha,\beta,\gamma]}(u) = M_n^{[\alpha,\beta,\gamma]} P_n^{[\alpha,\beta,\gamma]}(u)$; then we have:*

$$\mathcal{P}_n^{[\alpha,\beta,\gamma]''}(1) = (n^2 - 3n + 2)M_n^{[\alpha,\beta,\gamma]} - 2(n-1)[t^{n+2}]f^{[\alpha,\beta,\gamma]}(t) + [t^{n+2}]g^{[\alpha,\beta,\gamma]}(t)$$

where $f^{[\alpha,\beta,\gamma]}(t)$ is the function defined in Theorem 3.6 and

$$g^{[\alpha,\beta,\gamma]}(t) = \frac{2(\alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^3}{\alpha\gamma(1 - \alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))^2}.$$

Theorem 3.7 allows us to find a formula for computing the variance; in fact, we have:

$$\sigma_n^{[\alpha,\beta,\gamma]^2} = P_n^{[\alpha,\beta,\gamma]''}(1) + P_n^{[\alpha,\beta,\gamma]'}(1) - P_n^{[\alpha,\beta,\gamma]'}(1)^2 = \frac{\mathcal{P}_n^{[\alpha,\beta,\gamma]''}(1)}{M_n} + \mu_n^{[\alpha,\beta,\gamma]} - \mu_n^{[\alpha,\beta,\gamma]^2}$$

and, consequently, after some simplification:

$$\sigma_n^{[\alpha,\beta,\gamma]^2} = \frac{[t^{n+2}]g^{[\alpha,\beta,\gamma]}(t)}{M_n^{[\alpha,\beta,\gamma]}} - \frac{[t^{n+2}]f^{[\alpha,\beta,\gamma]}(t)}{M_n^{[\alpha,\beta,\gamma]}} - \left(\frac{[t^{n+2}]f^{[\alpha,\beta,\gamma]}(t)}{M_n^{[\alpha,\beta,\gamma]}}\right)^2 \tag{3.5}$$

When $\beta = 0$, by using formula (3.3) we obtain, after some simplification:

$$\sigma_{2m}^{[\alpha,0,\gamma]^2} = \sigma_{2m}^{[\alpha,\beta=0,\gamma]^2} - \frac{(m+1)(13m^2 - 27m + 14)}{4(m+2)(2m-1)^2} \tag{3.6}$$

According to (3.2), (3.4), (3.5) and (3.6), in Table 3.4 we give some values of $\mu_n^{[\alpha,\beta,\gamma]}$ and $\sigma_n^{[\alpha,\beta,\gamma]^2}$ for different α, β and γ and $4 \le n \le 8$.

n	4	5	6	7	8
$\mu_n^{[1,1,1]}$	26/9	27/7	245/51	733/127	2177/323
$\sigma_n^{[1,1,1]^2}$	8/81	6/49	512/2601	3858/16129	30412/104329
$\mu_n^{[1,0,1]}$	2	–	19/5	–	79/14
$\sigma_n^{[1,0,1]^2}$	0	–	4/25	–	73/196
$\mu_n^{[1,2,2]}$	53/18	133/34	649/133	521/89	3737/547
$\sigma_n^{[1,2,2]^2}$	17/324	93/1156	2138/17689	3676/23763	56082/299209
$\mu_n^{[1,2,3]}$	309/106	877/226	9671/1999	26349/4537	142544/21023
$\sigma_n^{[1,2,3]^2}$	873/11236	5373/51076	650646/3996001	4178664/20584369	108668520/441966529

Table 3.4: Some particular values for $\mu_n^{[\alpha,\beta,\gamma]}$ and $\sigma_n^{[\alpha,\beta,\gamma]^2}$.

In order to extract the coefficients in formulas (3.2) and (3.5), when $\beta \neq 0$, we need to know the dominant singularities of functions $f^{[\alpha,\beta,\gamma]}(t)$ and $g^{[\alpha,\beta,\gamma]}(t)$ (see, e.g., [27] for the asymptotic evaluation of the generating function coefficients).

Theorem 3.8 *The dominant singularity of the functions* $f^{[\alpha,\beta,\gamma]}(t)$ *and* $g^{[\alpha,\beta,\gamma]}(t)$ *coincides with the dominant singularity of* $d^{[\alpha,\beta,\gamma]}(t)$*, that is, with*

$$s^{[\alpha,\beta,\gamma]} = \frac{1}{\beta + 2\sqrt{\alpha\gamma}}.$$

Note: *when* $\beta = 0$ *the singularity has multiplicity 2.*

Proof: By substituting the expression for $d^{[\alpha,\beta,\gamma]}(t)$ in $f^{[\alpha,\beta,\gamma]}(t)$ and rationalizing (possibly with the help of a system of symbolic computation like Maple), we obtain:

$$f^{[\alpha,\beta,\gamma]}(t) = \frac{P(t) - Q(t)\sqrt{1 - 2\beta t + (\beta^2 - 4\alpha\gamma)t^2}}{2\alpha\gamma t(\alpha\gamma t + \beta)}$$

where

$$P(t) = 1 - 3\beta t - 4\alpha\gamma t^2 + 3\beta^2 t^2 + 5\alpha\beta\gamma t^3 - \beta^3 t^3 - \alpha\beta^2\gamma t^4 + 2\alpha^2\gamma^2 t^4$$

and

$$Q(t) = 1 - 2\beta t - 2\alpha\gamma t^2 + \beta^2 t^2 + \alpha\beta\gamma t^3.$$

Therefore, $f^{[\alpha,\beta,\gamma]}(t)$ has the three singularities $s_0 = 1/(\beta+2\sqrt{\alpha\gamma})$, $s_1 = 1/(\beta-2\sqrt{\alpha\gamma})$ and $s_2 = -\beta/(\alpha\gamma)$. The dominant singularity has to be chosen between s_0 and s_2 and we have:

$$\frac{\beta}{\alpha\gamma} = |s_2| \le |s_0| = \frac{1}{\beta + 2\sqrt{\alpha\gamma}}, \quad \Leftrightarrow \quad \beta^2 + 2\beta\sqrt{\alpha\gamma} - \alpha\gamma \le 0.$$

In other words, when $\alpha\gamma - \beta^2 \ge 2\beta\sqrt{\alpha\gamma}$ then s_2 is the dominant singularity; in this case, since $\beta\sqrt{\alpha\gamma} \ge 0$, we have $\alpha\gamma - \beta^2 \ge 0$. Moreover, we have,

$$P(s_2) = 1 - \frac{\beta^2}{\alpha\gamma}, \quad Q(s_2) = 1$$

and

$$\sqrt{1 - 2\beta s_2 + (\beta^2 - 4\alpha\gamma)s_2^2} = \sqrt{\frac{(\alpha\gamma - \beta^2)^2}{\alpha^2\gamma^2}} = \begin{cases} \frac{\alpha\gamma - \beta^2}{\alpha\gamma} & \text{if} \quad \alpha\gamma - \beta^2 \ge 0 \\ \frac{\beta^2 - \alpha\gamma}{\alpha\gamma} & \text{if} \quad \alpha\gamma - \beta^2 < 0 \end{cases}.$$

Now, if we set $R(t) = P(t) - Q(t)\sqrt{1 - 2\beta t + (\beta^2 - 4\alpha\gamma)t^2}$ then when $\alpha\gamma - \beta^2 \ge 0$ we can easily prove that $R(s_2) = 0$ and, therefore, the singularity s_2 can be eliminated. This proves that s_0 is always the dominant singularity of $f^{[\alpha,\beta,\gamma]}(t)$. Since $g^{[\alpha,\beta,\gamma]}(t) = 2f^{[\alpha,\beta,\gamma]}(t)/(1 - \alpha\gamma t^2 d^{[\alpha,\beta,\gamma]}(t))$, a similar reasoning yields the same result for $g^{[\alpha,\beta,\gamma]}(t)$. ■

We conclude with the following asymptotic results:

Theorem 3.9 *The average length* $\mu_n^{[\alpha,\beta,\gamma]}$ *of the relevant prefixes in coloured Motzkin words of length* n *satisfies*

$$\mu_n^{[\alpha,\beta,\gamma]} \sim n - 1 - \frac{\sqrt{\alpha\gamma}(3\beta\sqrt{\alpha\gamma} + 4\alpha\gamma)}{(\beta + \sqrt{\alpha\gamma})^2(\beta + 2\sqrt{\alpha\gamma})}, \qquad \beta \neq 0$$

with variance

$$\sigma_n^{[\alpha,\beta,\gamma]2} \sim \frac{3\alpha\gamma\beta^4 + 58\alpha^2\beta^2\gamma^2 + 24\alpha^{3/2}\beta^3\gamma^{3/2} + 54\alpha^{5/2}\beta\gamma^{5/2} + 16\alpha^3\gamma^3}{(\beta + \sqrt{\alpha\gamma})^4(\beta + 2\sqrt{\alpha\gamma})^2}, \qquad \beta \neq 0.$$

Proof: By developing $f^{[\alpha,\beta,\gamma]}(t)$ around its dominant singularity and putting $w = 1 - t/s^{[\alpha,\beta,\gamma]}$ we find: (omitting the constant term):

$$f^{[\alpha,\beta,\gamma]}(t) = -\frac{3\beta\sqrt{\alpha\gamma} + 4\alpha\gamma}{(\alpha\gamma)^{1/4}(\beta + \sqrt{\alpha\gamma})^2(\beta + 2\sqrt{\alpha\gamma})^{3/2}}\sqrt{w} + O(w).$$

Therefore,

$$[t^{n+2}]f^{[\alpha,\beta,\gamma]}(t) \sim -\frac{3\beta\sqrt{\alpha\gamma}+4\alpha\gamma}{(\alpha\gamma)^{1/4}(\beta+\sqrt{\alpha\gamma})^2(\beta+2\sqrt{\alpha\gamma})^{3/2}}\binom{1/2}{n+2}\left(\frac{-1}{s^{[\alpha,\beta,\gamma]}}\right)^{n+2}$$

and since the principal term of formula (2.4) in Theorem 2.1 gives

$$M_n^{[\alpha,\beta,\gamma]} \sim -\frac{(\beta+2\sqrt{\alpha\gamma})^{3/2}}{(\alpha\gamma)^{3/4}}\binom{1/2}{n}\left(\frac{-1}{s^{[\alpha,\beta,\gamma]}}\right)^{n},$$

we finally have

$$\frac{[t^{n+2}]f^{[\alpha,\beta,\gamma]}(t)}{M_n^{[\alpha,\beta,\gamma]}} \sim \frac{\sqrt{\alpha\gamma}(3\beta\sqrt{\alpha\gamma}+4\alpha\gamma)}{(\beta+\sqrt{\alpha\gamma})^2(\beta+2\sqrt{\alpha\gamma})^3}\left(\frac{-1}{s^{[\alpha,\beta,\gamma]}}\right)^2\frac{4n^2-1}{4n^2+12n+8} =$$

$$= \frac{\sqrt{\alpha\gamma}(3\beta\sqrt{\alpha\gamma}+4\alpha\gamma)}{(\beta+\sqrt{\alpha\gamma})^2(\beta+2\sqrt{\alpha\gamma})^3 {s^{[\alpha,\beta,\gamma]}}^2} + O\left(\frac{1}{n}\right) \sim \frac{\sqrt{\alpha\gamma}(3\beta\sqrt{\alpha\gamma}+4\alpha\gamma)}{(\beta+\sqrt{\alpha\gamma})^2(\beta+2\sqrt{\alpha\gamma})}$$

On the other hand, by developing $g^{[\alpha,\beta,\gamma]}(t)$ around the same singularity, we have

$$g^{[\alpha,\beta,\gamma]}(t) = -\frac{2(\alpha\gamma)^{1/4}(5\sqrt{\alpha\gamma}+4\alpha\gamma)}{(\beta+\sqrt{\alpha\gamma})^3(\beta+2\sqrt{\alpha\gamma})^{1/2}}\sqrt{w}+O(w)$$

and hence:

$$\frac{[t^{n+2}]g^{[\alpha,\beta,\gamma]}(t)}{M_n^{[\alpha,\beta,\gamma]}} \sim \frac{2\alpha\gamma(5\sqrt{\alpha\gamma}+3\alpha\gamma)}{(\beta+\sqrt{\alpha\gamma})^3(\beta+2\sqrt{\alpha\gamma})^2}\left(\frac{-1}{s^{[\alpha,\beta,\gamma]}}\right)^2\frac{4n^2-1}{4n^2+12n+8} \sim$$

$$\sim \frac{2\alpha\gamma(5\sqrt{\alpha\gamma}+3\alpha\gamma)}{(\beta+\sqrt{\alpha\gamma})^3}.$$

Finally, by substituting the previous results in formula (3.5) we obtain the value for ${\sigma_n^{[\alpha,\beta,\gamma]}}^2$ given in the statement of the theorem (the use of Maple at this stage becomes essential). ■

In particular, Theorem 3.9 in the case $\alpha=\beta=\gamma$ gives:

$$\mu_n^{[\alpha,\alpha,\alpha]} \sim n-1-\frac{7}{12} = n-\frac{19}{12}, \quad {\sigma_n^{[\alpha,\alpha,\alpha]}}^2 \sim \frac{155}{144},$$

a result which generalizes the analysis in [14, Theorem 6].

The previous theorem gives formulas for the average length of the relevant prefixes and the corresponding variance in the case $\beta \neq 0$. On the other hand, when $\beta = 0$, we have:

$$f^{[\alpha,0,\gamma]}(t) = \frac{1-4\alpha\gamma t^2+2\alpha^2\gamma^2t^4-(1-2\alpha\gamma t^2)\sqrt{1-4\alpha\gamma t^2}}{2\alpha^2\gamma^2t^2}$$

$$g^{[\alpha,0,\gamma]}(t) = \frac{1-5\alpha\gamma t^2+5\alpha^2\gamma^2t^4-(1-3\alpha\gamma t^2+\alpha^2\gamma^2t^4)\sqrt{1-4\alpha\gamma t^2}}{\alpha^3\gamma^3t^4}$$

and the coefficients of these functions can be exactly extracted. In the present case, let us first consider the walks relative to $\alpha=\gamma=1$; when we pass to general α and γ, we observe that each walk arriving at $(2m,0)$ has to be counted with multiplicity $(\alpha\gamma)^m$. On the other hand, also the relevant prefixes have the same multiplicity. Therefore, we conclude that $\mu_{2m}^{[\alpha,0,\gamma]}$ and ${\sigma_{2m}^{[\alpha,0,\gamma]}}^2$ should not depend on α and γ and consequently our results should coincide with case 1 in [14, Theorem 5], dealing with the case $\alpha=\gamma=1$. In fact, we have:

Theorem 3.10 *The average length* $\mu_{2m}^{[\alpha,0,\gamma]}$ *of the relevant prefixes in coloured Dyck walks of length* $2m$ *satisfies*

$$\mu_{2m}^{[\alpha,0,\gamma]} = 2m - \frac{13}{4} + O\left(\frac{1}{m}\right)$$

with variance

$$\sigma_{2m}^{[\alpha,0,\gamma]^2} = \frac{51}{16} + O\left(\frac{1}{m}\right)$$

Proof: When n is odd, we have $[t^{n+2}]f^{[\alpha,0,\gamma]}(t) = 0$. When $n = 2m > 0$, we obtain:

$$[t^{2m+2}]f^{[\alpha,0,\gamma]}(t) = [t^{m+1}]\frac{1 - 4\alpha\gamma t + 2\alpha^2\gamma^2t^2 - (1 - 2\alpha\gamma t)\sqrt{1 - 4\alpha\gamma t}}{2\alpha^2\gamma^2 t} =$$

$$= \frac{1}{\alpha\gamma}[t^{m+1}]\frac{1 - \sqrt{1 - 4\alpha\gamma t}}{2\alpha\gamma t} - 2[t^m]\frac{1 - \sqrt{1 - 4\alpha\gamma t}}{2\alpha\gamma t} + \frac{1}{\alpha\gamma}[t^{m+1}](\alpha\gamma t - 1) =$$

$$= \frac{1}{\alpha\gamma}\frac{(\alpha\gamma)^{m+1}}{m+2}\binom{2m+2}{m+1} - 2\frac{(\alpha\gamma)^m}{m+1}\binom{2m}{m} = \frac{2(\alpha\gamma)^m(m-1)}{(m+1)(m+2)}\binom{2m}{m}.$$

Therefore, from formulae (3.2) and (3.4), we have:

$$\mu_{2m}^{[\alpha,0,\gamma]} = 2m - 1 - 2\frac{m-1}{m+2} - \frac{m+1}{2(2m-1)} = \frac{(m-1)(8m^2 + 7m + 2)}{2(2m-1)(m+2)} =$$

$$= 2m - \frac{13}{4} + \frac{6}{m+2} - \frac{3}{4(2m-1)}.$$

For what concerns the function $g^{[\alpha,0,\gamma]}(t)$, we obtain for $n = 2m > 0$:

$$[t^{2m+2}]g^{[\alpha,0,\gamma]}(t) = [t^{m+1}]\frac{1 - 5\alpha\gamma t + 5\alpha^2\gamma^2t^2 - (1 - 3\alpha\gamma t + \alpha^2\gamma^2t^2)\sqrt{1 - 4\alpha\gamma t}}{\alpha^3\gamma^3t^2} =$$

$$= \frac{2}{\alpha^2\gamma^2}[t^{m+2}]\frac{1 - \sqrt{1 - 4\alpha\gamma t}}{2\alpha\gamma t} - \frac{6}{\alpha\gamma}[t^{m+1}]\frac{1 - \sqrt{1 - 4\alpha\gamma t}}{2\alpha\gamma t} + 2[t^m]\frac{1 - \sqrt{1 - 4\alpha\gamma t}}{2\alpha\gamma t} =$$

$$= \frac{2}{\alpha^2\gamma^2}\frac{(\alpha\gamma)^{m+2}}{m+3}\binom{2m+4}{m+2} - \frac{6}{\alpha\gamma}\frac{(\alpha\gamma)^{m+1}}{m+2}\binom{2m+2}{m+1} + 2\frac{(\alpha\gamma)^m}{m+1}\binom{2m}{m} =$$

$$= \frac{10(\alpha\gamma)^m m(m-1)}{(m+1)(m+2)(m+3)}\binom{2m}{m}.$$

Therefore, from formula (3.6) we have:

$$\sigma_{2m}^{[\alpha,0,\gamma]^2} = \frac{10m(m-1)}{(m+2)(m+3)} - 2\frac{m-1}{m+2} - 4\frac{(m-1)^2}{(m+2)^2} - \frac{(m+1)(13m^2 - 27m + 14)}{4(m+2)(2m-1)^2} =$$

$$= \frac{3(m-1)(m-2)(17m^3 + 2m^2 - 21m - 14)}{4(m+2)^2(m+3)(2m-1)^2} =$$

$$= \frac{51}{16} + \frac{456}{5(m+2)} - \frac{120}{m+3} - \frac{39}{40(2m-1)} - \frac{36}{(m+2)^2} - \frac{9}{16(2m-1)^2}.$$

As expected, neither $\mu_{2m}^{[\alpha,0,\gamma]}$ nor $\sigma_{2m}^{[\alpha,0,\gamma]^2}$ depend on α and γ. ■

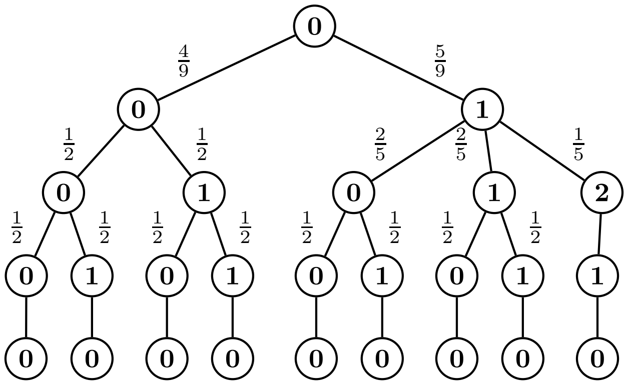

Figure 4.3: The tree for the generation of Motzkin walks of length $n = 4$ ($\alpha = \beta = \gamma = 1$).

4 The random generation

In this section we describe an algorithm which generates uniformly at random a coloured Motzkin walk of a given length n, that is, each coloured Motzkin walk is generated with probability $1/M_n^{[\alpha,\beta,\gamma]}$. As we noted in the Introduction, the method behind this algorithm goes back to Nijenhuis and Wilf [24] and the explicit use of generating trees recalls the approach of [2]. However, as far as we know, the application of the method to coloured Motzkin walks, for generic α, β, and γ, is new and the results of Section 3 provide an exact average case analysis of the corresponding algorithm, described in Table 4.7.

In order to explain the algorithm, let us consider the tree in Figure 4.3 which corresponds to the generating tree (2.1) with $\alpha = \beta = \gamma = 1$ and up to level $n = 4$. The branches which do not correspond to walks ending at altitude 0 have been deleted; each remaining branch is labeled with a value of probability which depends on the length of the random walk we want to generate. In the figure, for example, these probability values correspond to the generation of walks of length 4. At level 4 in the generating tree there are 9 nodes labeled 0: 4 are in the left subtree of the root while 5 are in the right subtree. Thus the first two branches are labeled with 4/9 and 5/9. In particular we observe that:

- 4/9 corresponds to the ratio between the number of nodes with label 0 at level 3 and the number of nodes with label 0 at level 4;
- 5/9 corresponds to the ratio between the number of nodes with label 0 at level 3 in the sub-tree having root with label 1, and the number of nodes with label 0 at level 4.

The other probability values are computed similarly and this reasoning can be applied to every n and every label k at level i. We denote by $\mathcal{T}_{(0)}^{[\alpha,\beta,\gamma]}$ the tree corresponding to the rule (2.1) and by $\mathcal{T}_{(j)}^{[\alpha,\beta,\gamma]}$ the tree corresponding to the specification (4.1)

$$\left\{ \begin{array}{lll} \mathit{root}: & (j) & \\ \mathit{rule}: & (k) & \rightarrow (k-1)^{\gamma}(k)^{\beta}(k+1)^{\alpha} \end{array} \right. \tag{4.1}$$

We have the following important result:

Theorem 4.1 *Let $f_{n,j}^{[\alpha,\beta,\gamma]}$ be the number of nodes labelled 0 at level n in the tree $\mathcal{T}_{(j)}^{[\alpha,\beta,\gamma]}$. Then we have:*

$$f_{n,j}^{[\alpha,\beta,\gamma]} = M_{n,j}^{[\gamma,\beta,\alpha]}.$$

Proof: The proof follows from a straightforward combinatorial reasoning. In fact, $f_{n,j}^{[\alpha,\beta,\gamma]}$ corresponds to the number of coloured walks of length n which start at altitude j and end at altitude 0 and this corresponds exactly to the number of walks of length n starting from 0 and ending in j, when the role of α and γ is exchanged. ■

Therefore, referring to Figure 4.3, if we fix n and consider a node with label k at level i, then we label the branches of the tree with probabilities $p^{[1,1,1]}_{n,k,i}$, $q^{[1,1,1]}_{n,k,i}$, $r^{[1,1,1]}_{n,k,i}$ such that:

$$p^{[1,1,1]}_{n,k,i} = \frac{f^{[1,1,1]}_{n-i-1,k-1}}{f^{[1,1,1]}_{n-i,k}}, \quad q^{[1,1,1]}_{n,k,i} = \frac{f^{[1,1,1]}_{n-i-1,k}}{f^{[1,1,1]}_{n-i,k}}, \quad r^{[1,1,1]}_{n,k,i} = \frac{f^{[1,1,1]}_{n-i-1,k+1}}{f^{[1,1,1]}_{n-i,k}} = 1 - (p^{[1,1,1]}_{n,k,i} + q^{[1,1,1]}_{n,k,i}).$$

In the general case with colours, the involved quantities are

$$p^{[\alpha,\beta,\gamma]}_{n,k,i} = \frac{f^{[\alpha,\beta,\gamma]}_{n-i-1,k-1}}{f^{[\alpha,\beta,\gamma]}_{n-i,k}}, \quad q^{[\alpha,\beta,\gamma]}_{n,k,i} = \frac{f^{[\alpha,\beta,\gamma]}_{n-i-1,k}}{f^{[\alpha,\beta,\gamma]}_{n-i,k}}, \quad r^{[\alpha,\beta,\gamma]}_{n,k,i} = \frac{f^{[\alpha,\beta,\gamma]}_{n-i-1,k+1}}{f^{[\alpha,\beta,\gamma]}_{n-i,k}},$$

which, due to formula (2.3), satisfy the relation:

$$\gamma p^{[\alpha,\beta,\gamma]}_{n,k,i} + \beta q^{[\alpha,\beta,\gamma]}_{n,k,i} + \alpha r^{[\alpha,\beta,\gamma]}_{n,k,i} = 1.$$

In this case, we consider a tree in which each node has a number of children which depends on the number of colours and where each branch corresponding to the same step, but with different colour, is marked with the same value of probability. In Figure 4.4, for example, we give the tree for the generation of coloured Motzkin walks of length $n = 3$, in the case $\beta = \gamma = 2$ and $\alpha = 1$. As it can be easily checked with Tables 4.5 and 4.6, we have, for example:

n/k	0	1	2	3	4	5	6	7
0	1							
1	2	1						
2	6	4	1					
3	20	16	6	1				
4	72	64	30	8	1			
5	272	260	140	48	10	1		
6	1064	1072	636	256	70	12	1	
7	4272	4480	2856	1288	420	96	14	1

Table 4.5: The triangle $M^{[1,2,2]}$

n/k	0	1	2	3	4	5	6	7
0	1							
1	2	2						
2	6	8	4					
3	20	32	24	8				
4	72	128	120	64	16			
5	272	520	560	384	160	32		
6	1064	2144	2544	2048	1120	384	64	
7	4272	8960	11424	10304	6720	3072	896	128

Table 4.6: The triangle $M^{[2,2,1]}$

$$q^{[1,2,2]}_{3,0,0} = \frac{f^{[1,2,2]}_{2,0}}{f^{[1,2,2]}_{3,0}} = \frac{3}{10}, \quad r^{[1,2,2]}_{3,0,0} = \frac{f^{[1,2,2]}_{2,1}}{f^{[1,2,2]}_{3,0}} = \frac{2}{5},$$

$$q^{[1,2,2]}_{3,0,1} = \frac{f^{[1,2,2]}_{1,0}}{f^{[1,2,2]}_{2,0}} = r^{[1,2,2]}_{3,0,1} = \frac{f^{[1,2,2]}_{1,1}}{f^{[1,2,2]}_{2,0}} = \frac{1}{3}, \quad p^{[1,2,2]}_{3,1,1} = \frac{f^{[1,2,2]}_{1,0}}{f^{[1,2,2]}_{2,1}} = q^{[1,2,2]}_{3,1,1} = \frac{f^{[1,2,2]}_{1,1}}{f^{[1,2,2]}_{2,1}} = \frac{1}{4}.$$

The algorithm for the random generation of a coloured Motzkin walk is described in Table 4.7 in `Maple-like` style. We use two routines for random number generation: `random()` returns a real number λ, with $0 \leq \lambda < 1$, and `random(h)` returns an integer number between 1 and h. We code the colours with integer numbers in the range $[1 \dots \alpha]$, $[1 \dots \beta]$ and $[1 \dots \gamma]$, respectively. The algorithm performs a number of calls to `random()` and moves along the branches of the tree according to the values of the corresponding cumulated probabilities. In particular, the number λ generated by `random()` determines the step and the corresponding colour in the following way: if $0 \leq \lambda < \gamma p_{n,k,i}^{[\alpha,\beta,\gamma]}$ then we have a down step; if $\gamma p_{n,k,i}^{[\alpha,\beta,\gamma]} \leq \lambda < \gamma p_{n,k,i}^{[\alpha,\beta,\gamma]} + \beta q_{n,k,i}^{[\alpha,\beta,\gamma]}$ then we have an horizontal step; otherwise we have an up step. The colours are determined by considering them as uniformly distributed in the corresponding range of values. In particular, the colour is computed by

$$\left\lfloor \frac{\lambda}{p_{n,k,i}^{[\alpha,\beta,\gamma]}} \right\rfloor + 1, \qquad \left\lfloor \frac{\lambda - \gamma p_{n,k,i}^{[\alpha,\beta,\gamma]}}{q_{n,k,i}^{[\alpha,\beta,\gamma]}} \right\rfloor + 1, \qquad \left\lfloor \frac{\lambda - \gamma p_{n,k,i}^{[\alpha,\beta,\gamma]} - \beta q_{n,k,i}^{[\alpha,\beta,\gamma]}}{r_{n,k,i}^{[\alpha,\beta,\gamma]}} \right\rfloor + 1$$

for down, horizontal and up steps, respectively. As soon as the relevant prefix of the walk is reached, we need to generate the remaining steps as down steps; a possible exception is the last step which can be horizontal when the relevant prefix has length $n-1$. The algorithm returns a sequence of ordinates $y = (y_1, y_2, \cdots, y_n)$ and a sequence of colours $c = (c_1, c_2, \cdots, c_n)$ denoting the walk $((0,0), (1, y_1), \cdots, (n, y_n))$ having the i-th step of colour c_i. The colours are randomly chosen according to the kind of step. In Table 4.7 a comma denotes the concatenation of an element to a sequence.

Input: the length $n \geq 1$ of the walk; the number of colours α, β, γ;
Output: two sequences $y = (y_1, \cdots, y_n)$ and $c = (c_1, \cdots, c_n)$.

```
$y$ :=NULL; $c$ :=NULL; $k := 0$;
finished:=false;
for i from 0 to $n-2$ while not finished do
    $\lambda$:=random(); $p := 0$; colour:= 0;
    if $k > 0$ then
        $p := M_{n-i-1,k-1}^{[\gamma,\beta,\alpha]} / M_{n-i,k}^{[\gamma,\beta,\alpha]}$;
        if $\lambda \leq \gamma p$ then $k := k-1$; colour:= $\lfloor \frac{\lambda}{p} \rfloor + 1$; fi;
    fi;
    if colour= 0 then
        $q := M_{n-i-1,k}^{[\gamma,\beta,\alpha]} / M_{n-i,k}^{[\gamma,\beta,\alpha]}$;
        if $\lambda \leq \gamma p + \beta q$ then $k := k$; colour:= $\lfloor \frac{\lambda - \gamma p}{q} \rfloor + 1$; fi;
    fi;
    if colour= 0 then $k := k+1$; $r := (1 - \gamma p - \beta q)/\alpha$; colour:= $\lfloor \frac{\lambda - \gamma p - \beta q}{r} \rfloor + 1$; fi;
    $y := y, k$;
    $c := c$, colour;
    if $k = n - i - 1$ then
        finished:=true;
        $y := y, \mathbf{seq}(k - j, j = 1..k-1)$;
        $c := c, \mathbf{seq}(\mathbf{random}(\gamma), j = 1..k-1)$;
    fi;
od;
$y := y, 0$;
if $y_{n-1} = 1$ then $c := c, \mathbf{random}(\gamma)$ else $c := c, \mathbf{random}(\beta)$ fi
return $y, c$;
```

Table 4.7: Algorithm for the random generation of a coloured Motzkin walk; λ is a real number with $0 \leq \lambda < 1$.

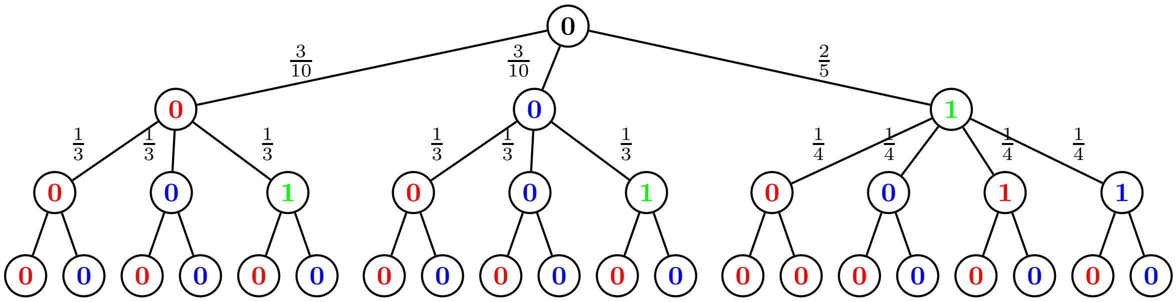

Figure 4.4: The tree for the generation of coloured Motzkin walks of length $n = 3$ ($\beta = \gamma = 2$, $\alpha = 1$).

Particular attention has to be paied when $\beta = 0$. In Figure 4.5 we illustrate the tree for the generation of Dyck walks ($\alpha = \gamma = 1$ and $\beta = 0$). In this case, the algorithm could be slightly improved taking into account the facts that the relevant prefix of a coloured Dyck walk has length at most $n - 2$ and that when the walk reaches $y = 0$, then the next step has to arrive at $y = 1$.

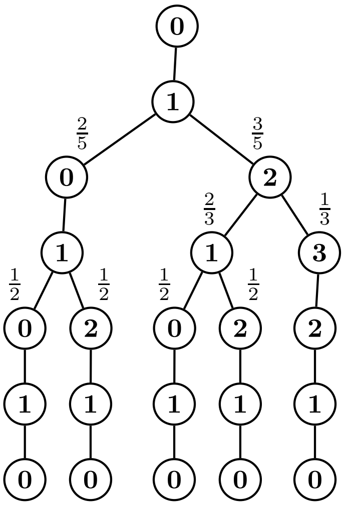

Figure 4.5: The tree for the generation of Dyck walks of length $n = 6$ ($\alpha = \gamma = 1$, $\beta = 0$).

In conclusion, it is quite obvious that the average number of calls to `random()` (the calls which require the computation of some values of probability) corresponds to the average length of the relevant prefixes of coloured Motzkin walks (more precisely, when $\beta = 0$, the number of calls to `random` is bounded by this quantity). Therefore, by using the results of Section 3, we have the following:

Corollary 4.2 *The average number of calls to `random()` performed by the algorithm described in Table 4.7 and the corresponding variance are given by the formulas for $\mu_n^{[\alpha,\beta,\gamma]}$ and ${\sigma_n^{[\alpha,\beta,\gamma]}}^2$ in Theorem 3.9, if $\beta \neq 0$. The average number of calls to random is bounded by the formula for $\mu_{2m}^{[\alpha,0,\gamma]}$ in Theorem 3.10, when $\beta = 0$.*

Finally, we wish to point out that algorithm in Table 4.7, as the algorithm for ranking presented in [14], is based on the computation of values in the coloured Motzkin triangle. Therefore, the average complexity of these procedures is bounded by the average length of the relevant prefixes provided that the computation of these numbers is performed in constant amount of time. This can be realized by a precomputation of the triangle.

5 Conclusions

In the present paper we have performed an analysis of the length of the relevant prefixes of coloured Motzkin words where a relevant prefix is the minimal sub-walk such that there is exactly one continuation to a coloured Motzkin walk. By using the concepts of probability generating functions and Riordan arrays, we have been able to compute the average length and the corresponding variance in terms of the numbers α, β and γ of colours used for the three kinds of steps. The problem has been solved also for coloured Dyck walks which correspond to the particular case $\beta = 0$. This result represents and advance in the combinatorial analysis of coloured Motzkin walks and gives the average complexity of algorithms for the random generation, ranking and unranking of coloured Motzkin walks, based on a tree structure encoding the walks.

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