An algebraic-combinatorial approach for studying coloured Dyck-Schröder paths

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Abstract

We characterize coloured Dyck and Schröder paths in both an algebraic and combinatorial way. In fact, we give algebraic and combinatorial proofs that, starting from the definition of such paths, we obtain a generating function and, from this, the corresponding recurrence. Finally, by using a generalization of a beautiful bijection of Sulanke [5], we give a combinatorial proof of this recurrence, thus closing the ideal cycle connecting all the basic properties of the combinatorial objects considered.

1 Introduction

In this paper we consider a class $\mathcal{S}$ of coloured lattice paths in $\mathbf{Z}^2$ which start at the origin $(0,0)$ and arrive to the point $(2n,0)$, $n \geq 1$; they never touch or pass below the x-axis, except for their starting and ending points and are made up of the following three kinds of steps:

- $(1,1)$, or *up* steps, going from a point (x,y) to $(x+1,y+1)$;
- $(2,0)$, or *double horizontal* steps, going from a point (x,y) to $(x+2,y)$;
- $(1,-1)$, or *down* steps, going from a point (x,y) to $(x+1,y-1)$.

These steps can have different colours and, in particular, we study the parametric case in which there are $a>0$ colours for up steps, $b \geq 0$ colours for double horizontal steps and $c>0$ colours for down steps. The set of all paths arriving to $(2n,0)$ will be denoted by $\mathcal{S}_n$, the corresponding cardinality by $S_n = |\mathcal{S}_n|$, and their generating function by $S(t) = \sum_{n\geq 0} S_n t^n$. If we ignore the first up step and the last down step, we obtain the set $\overline{\mathcal{S}}$ of coloured paths that never pass below the x-axis, but possibly touch it at intermediate points. Paths of this kind starting at the origin and arriving to $(2n,0)$ will be denoted by $\overline{\mathcal{S}}_n$, with $\overline{S}_n$ denoting the cardinality $|\overline{\mathcal{S}}_n|$ and with $\overline{S}(t) = \sum_{n\geq 0} \overline{S}_n t^n$ denoting the corresponding generating function.

Some cases are well-known in the literature; for example, when $a=c=1$ and $b=0$ we obtain the so-called *elevated Dyck paths*, which are counted by the Catalan numbers $S_n = \{0,1,1,2,5,14,\cdots\}$. When $a=b=c=1$, we obtain the *elevated Schröder paths*, counted by the big Schröder numbers $S_n = \{0,1,2,6,22,90,\cdots\}$, $n \geq 0$. Actually, these two situations are the most important, but some coloured versions of these paths have some interesting combinatorial intrepretations. For example, if we consider Schröder paths in which the up steps can have a different colours, double horizontal steps can have b different colours and down steps can have only one colour ($c=1$), we obtain a way to count the algebraic expressions according to the number a of different binary operators

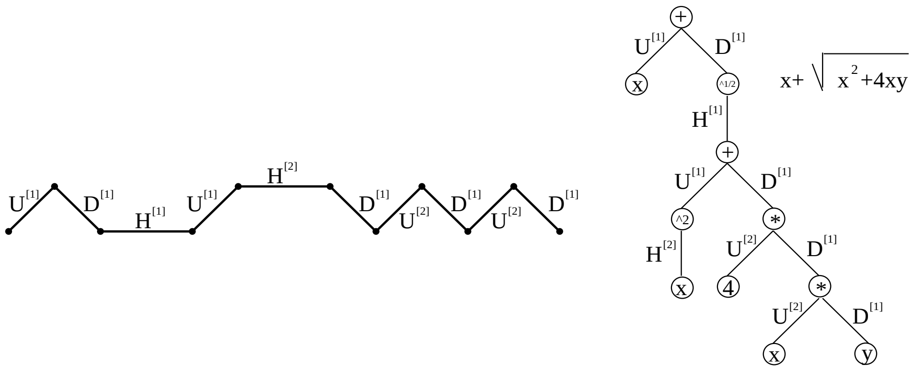

Figure 1.1: Coloured Schröder paths and algebraic expressions.

and the number b of different unary operators (see Figure 1.1). In any case, we wish to point out that our approach gives a unified treatment of Dyck and Schröder paths and our algebraic and combinatorial proofs work for any kind of these paths.

The aim of this paper is threefold:

- to find the properties of these generalized (elevated) Dyck and Schröder paths;
- to extend a bijective proof first found by Sulanke [5] to all these kinds of paths;
- to show that these paths constitute an example of a complete "algebraico-combinatorial" object.

In order to explain this last sentence, let us consider the common way to approach a combinatorial problem. When we wish to study some combinatorial object, we often try to find a recurrence relation describing the whole set of objects; then we pass from the recurrence to a generating function and finally use the generating function to derive properties of the original objects. In this way combinatorial and algebraic considerations and arguments are mixed together and used according to the preference of the author and/or their convenience in the particular case. Some authors prefer a purely combinatorial approach, and only use algebra when strictly necessary or when they are not able to find a suitable combinatorial proof. Algebraic arguments are usually more simple and they are often considered of less value because they only give a formal solution and do not allow us to see the "deep" and structural aspects of the problem; on the other hand, algebra can never be completely avoided, because, at some point in our study, we should arrive at some counting results. In principle, all the properties of a combinatorial object should be characterized in both an algebraic *and* a combinatorial way, but this is not always possible, and some properties are proved by combinatorial arguments, while others by algebraic reasoning.

A complete "algebraico-combinatorial" object is a combinatorial object whose properties are proved both by algebraic and combinatorial arguments. We show that our paths are, essentially, objects of this type and that they satisfy the *cycle* described in Figure 1.2. In particular:

- we use combinatorial considerations to obtain a formal description of the paths through a context-free grammar;
- we apply the Schützenberger methodology to pass from the description to the generating functions (algebraic approach);

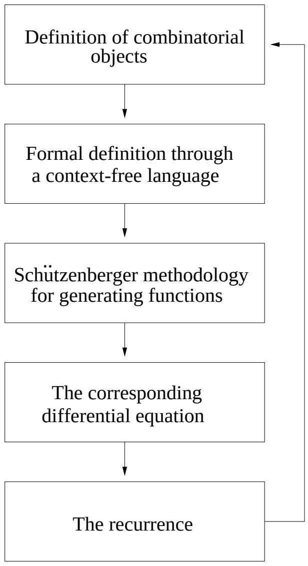

Figure 1.2: The *cycle* of an “algebraico-combinatorial” object.

- we find the differential equation satisfied by the generating function, so that we can derive the recurrence relations satisfied by our objects in an algebraic fashion; we use the method of “indeterminate coefficients” applied to differential equations.
- by a generalization of Sulanke bijection [5], we prove in a combinatorial way, that the recurrence relation is satisfied by our original objects.

We observe that these arguments can be “reversed”, that is, proved in the other direction, or, if one prefers, the arrows in the cycle can be double headed.

2 Toward the recurrence relation

Let $\mathcal{S} = \cup_n \mathcal{S}_n$ be the set of all paths as defined in the Introduction. Let $U^{[a]}$ $H^{[b]}$ and $D^{[c]}$ be the three sets of up, double horizontal and down steps. A path in $\mathcal{S}$ starts with an up step, which can assume a different colours, and ends with a down step, which, in turn, can assume c different colours. By using standard notation for context-free grammars (BNF), we write:

$$\mathcal{S} ::= U^{[a]}\overline{\mathcal{S}}D^{[c]} \tag{2.1}$$

where $\overline{\mathcal{S}}$ denotes the set of paths that run from $(0,0)$ to $(2n,0)$, for $n \geq 1$, (that never pass below the x-axis). We have:

$$\overline{\mathcal{S}} ::= \epsilon \mid H^{[b]}\overline{\mathcal{S}} \mid U^{[a]}\overline{\mathcal{S}}D^{[c]}\overline{\mathcal{S}}, \tag{2.2}$$

by distinguishing among an empty path, a path which starts with any of the $H^{[b]}$ double horizontal steps and a path which starts with whatever $U^{[a]}$ step. By applying Schützenberger methodology [4] to the grammar defining $\overline{\mathcal{S}}$ we get the following equation in $\overline{S}(t) = \sum_{n\geq 0} \overline{S}_n t^n$:

$$\overline{S}(t) = 1 + bt\overline{S}(t) + act\overline{S}(t)^2.$$

Since $\overline{S}(0) = |\epsilon| = 1$, the appropriate solution to this second-degree equation is:

$$\overline{S}(t) = \frac{1 - bt - \sqrt{1 - 2(b+2ac)t + b^2t^2}}{2act}.$$

If we let $S(t) = \sum_{n\geq 0} S_n t^n$, we obviously get $S(t) = act\overline{S}(t)$, hence

$$S(t) = \frac{1 - bt - \sqrt{1 - 2(b + 2ac)t + b^2t^2}}{2}.$$

We have thus found, by using the first three steps illustrated in Figure 1.2, the counting generating function for class $\mathcal{S}$.

Before proceeding to close the cycle, we next examine some properties characterizing our coloured paths. The first property we consider is related to the *Narayana numbers* $N_{n,j} = \frac{1}{n}\binom{n}{j-1}\binom{n}{j}$, $1 \leq j \leq n$. These numbers are well known in the literature since they have many combinatorial interpretations (see for example Sulanke [6] which describes many properties of Dyck paths having the Narayana distribution). In particular, Narayana numbers $N_{n,j}$ count the number of (not elevated) Dyck paths of semilength n having exactly j *peaks* (in the sequel, we will take this result as granted). Figure 2.3, in which a marked point denotes a peak, illustrates the peaks' distribution for $n = 3$: we have $N_{3,1} = 1$ path with 1 peak, $N_{3,2} = 3$ paths with 2 peaks and $N_{3,3} = 1$ path with 3 peaks.

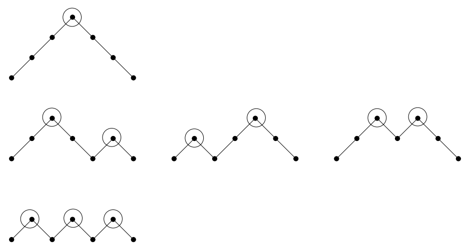

Figure 2.3: The peaks' distribution in Dyck paths of semilength 3.

Here, we are interested in *Narayana polynomials* as defined in Sulanke [6]:

$$N_0(w) = 1, \qquad N_n(w) = \sum_{j=1}^{n} N_{n,j} w^j, \quad n > 0$$

We remark that the paper [1] analyzed the n^{th} "Narayana polynomial" defined differently as $\sum_{j=1}^{n} N_{n,j}(w+1)^j = N_n(w+1)$. We can prove the following theorem, both in an algebraic and a combinatorial way:

Theorem 2.1 *Class $\mathcal{S}$ is related to the Narayana polynomials by the following formula:*

$$S_n = (ac)^n N_{n-1}\left(\frac{b + ac}{ac}\right), \quad n \geq 1.$$

The algebraic proof: We first consider the ϵ-free grammar we obtain from (2.2):

$$\overline{\mathcal{S}}^{\epsilon} ::= H^{[b]} \mid H^{[b]}\overline{\mathcal{S}}^{\epsilon} \mid U^{[a]}D^{[c]} \mid U^{[a]}D^{[c]}\overline{\mathcal{S}}^{\epsilon} \mid U^{[a]}\overline{\mathcal{S}}^{\epsilon}D^{[c]} \mid U^{[a]}\overline{\mathcal{S}}^{\epsilon}D^{[c]}\overline{\mathcal{S}}^{\epsilon} \tag{2.3}$$

and let $\overline{S}^{\epsilon}(t) = \sum_{n\geq 0} \overline{S}^{\epsilon}_n t^n$ where $\overline{S}^{\epsilon}_n = |\overline{\mathcal{S}}^{\epsilon}|$. We obviosly have $\overline{S}(t) = \overline{S}^{\epsilon}(t) + 1$. By applying Schützenberger methodology to (2.3) we get:

$$\overline{S}^{\epsilon}(t) = t(b(1 + \overline{S}^{\epsilon}(t)) + ac(1 + \overline{S}^{\epsilon}(t))^2) = t(1 + \overline{S}^{\epsilon}(t))(b + ac + ac\overline{S}^{\epsilon}(t)) = t\Phi(\overline{S}^{\epsilon}(t)).$$

We are in the hypotheses of the Lagrange Inversion Theorem:

$$\overline{S}^{\epsilon}_n = [t^n]\overline{S}^{\epsilon}(t) = \frac{1}{n}[w^{n-1}]\Phi(w)^n = \frac{1}{n}[w^{n-1}](1 + w)^n \left(1 + \frac{ac}{b + ac}w\right)^n.$$

This is a convolution, and we get:

$$\overline{S}_n^{\epsilon} = \frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k}\binom{n}{k+1}\left(\frac{b+ac}{ac}\right)^{k+1}(ac)^n =$$

$$= (ac)^n \sum_{j=1}^{n} \frac{1}{n}\binom{n}{j-1}\binom{n}{j}\left(\frac{b+ac}{ac}\right)^{j} = (ac)^n \sum_{j=1}^{n} N_{n,j}\left(\frac{b+ac}{ac}\right)^{j} = (ac)^n N_n\left(\frac{b+ac}{ac}\right).$$

Moreover,

$$S_n = [t^n]S(t) = ac[t^{n-1}](1+\overline{S}^{\epsilon}(t)) = (ac)^n N_{n-1}\left(\frac{b+ac}{ac}\right).$$

The combinatorial proof: Let us consider the $N_{n,j}$ *Dyck* paths of semilength n and having j peaks. Every peak can either be transformed into a double horizontal step and therefore be coloured by b different colours or remain as it is assuming ac different colours, a colors for the up step combined with c colours of the down step. In conclusion, it may assume $b+ac$ different forms and/or colours. Then, $2n-2j$ up and down steps remain, $n-j$ up and down steps respectively. The up steps can assume a different colours while the down steps c. Therefore, every one of the $N_{n,j}$ paths can be transformed into $(b+ac)^j a^{n-j} c^{n-j}$ different coloured (not elevated) Schröder paths. In all we have:

$$\overline{S}_n = \sum_{j=1}^{n} N_{n,j}(b+ac)^j a^{n-j} c^{n-j}$$

and in the notation above this is exactly $(ac)^n N_n\left(\frac{b+ac}{ac}\right)$. The proof follows from the relation $S_n = ac\overline{S}_{n-1}$. ■

The previous theorem is an example of how sometimes the combinatorial proof can be more direct than the algebraic proof.

We point out, in particular, two special cases. When $a=c=1$ and $b=0$, i.e. in the Catalan case, Theorem 2.1 becomes:

$$S_n = N_{n-1}(1) = \sum_{j=1}^{n-1} N_{n-1,j};$$

when $a=b=c=1$, i.e. in the Schröder case, Theorem 2.1 becomes:

$$S_n = N_{n-1}(2) = \sum_{j=1}^{n-1} N_{n-1,j}2^j.$$

The following theorem establishes the number of occurrences of step pairs of the form $D^{[c]}H^{[b]}$ or $H^{[b]}H^{[b]}$ on the totality of paths of $\mathcal{S}_n$:

Theorem 2.2 *For $n \geq 2$, there are $b(n-2)S_{n-1}$ step pairs of the form $D^{[c]}H^{[b]}$ or $H^{[b]}H^{[b]}$ on the totality of paths of $\mathcal{S}_n$.*

The algebraic proof: In the grammar (2.3) we can distinguish between paths that start with an $H^{[b]}$ step ($\overline{\mathcal{S}}^{H^{[b]}}$) and paths that start with a $U^{[a]}$ step ($\overline{\mathcal{S}}^{U^{[a]}}$):

$$\overline{\mathcal{S}}^{\epsilon} ::= \overline{\mathcal{S}}^{H^{[b]}} | \overline{\mathcal{S}}^{U^{[a]}}$$

$$\overline{\mathcal{S}}^{H^{[b]}} ::= H^{[b]} \mid H^{[b]}\overline{\mathcal{S}}^{H^{[b]}} \mid H^{[b]}\overline{\mathcal{S}}^{U^{[a]}}$$

$$\overline{\mathcal{S}}^{U^{[a]}} ::= U^{[a]}D^{[c]} \mid U^{[a]}D^{[c]}\overline{\mathcal{S}}^{H^{[b]}} \mid U^{[a]}D^{[c]}\overline{\mathcal{S}}^{U^{[a]}} \mid U^{[a]}\overline{\mathcal{S}}^{\epsilon}D^{[c]} \mid U^{[a]}\overline{\mathcal{S}}^{\epsilon}D^{[c]}\overline{\mathcal{S}}^{H^{[b]}} \mid U^{[a]}\overline{\mathcal{S}}^{\epsilon}D^{[c]}\overline{\mathcal{S}}^{U^{[a]}}$$

We now apply the Schützenberger methodology by introducing the new indeterminate w to mark the step pairs $D^{[c]}H^{[b]}$ or $H^{[b]}H^{[b]}$. We thus obtain the following system of equations:

$$\overline{S}^{\epsilon}(t,w) = \overline{S}^{H^{[b]}}(t,w) + \overline{S}^{U^{[a]}}(t,w)$$

$$\overline{S}^{H^{[b]}}(t,w) = t + t\overline{S}^{U^{[a]}}(t,w) + tw\overline{S}^{H^{[b]}}(t,w)$$

$$\overline{S}^{U^{[a]}}(t,w) = t + tw\overline{S}^{H^{[b]}}(t,w) + t\overline{S}^{U^{[a]}}(t,w) + t\overline{S}^{\epsilon}(t,w) + tw\overline{S}^{\epsilon}(t,w)\overline{S}^{H^{[b]}}(t,w) + t\overline{S}^{\epsilon}(t,w)\overline{S}^{U^{[a]}}(t,w)$$

being

$$\overline{S}(t,w) = 1 + \overline{S}^{\epsilon}(t,w) \quad \text{and} \quad S(t,w) = act\overline{S}(t,w) = \sum_{n,k} S_{n,k}t^n w^k$$

where $S_{n,k}$ denotes the number of paths in $\mathcal{S}_n$ having k occurrences of step pairs $D^{[c]}H^{[b]}$ or $H^{[b]}H^{[b]}$. Obviously, to prove the theorem, we have to find the value $\sum_{k\geq 1} kS_{n,k}$. By solving the previous system we get:

$$S(t,w) = \frac{1 - btw - \sqrt{1 - 2(bw + 2ac) + b(bw^2 + 4acw - 4ac)t^2}}{2}$$

which corresponds to $S(t)$ if we put $w = 1$. The sum $\sum_{k\geq 1} kS_{n,k}$ can be computed by differentiating $S(t,w)$ with respect to w and then putting $w = 1$:

$$\sum_{k\geq 1} kS_{n,k} = [t^n]\left.\frac{\partial S(t,w)}{\partial w}\right|_{w=1} = [t^n]\frac{bt(1 - bt - 2act - \sqrt{1 - 2(b+2ac)t + b^2t^2})}{2\sqrt{1 - 2(b+2ac)t + b^2t^2}} =$$

$$= [t^n]bt^2S'(t) - [t^n]btS(t) = b(n-2)S_{n-1}.$$

The combinatorial proof: We first consider uncoloured Schröder paths $\mathcal{S}_{n-1}$ ($a = b = c = 1$). If we ignore the last down step, there are exactly $n-2$ ending points of down and double horizontal steps, corresponding to the marked points of Figure 2.4, which illustrates case $n = 4$. If we add, beginning from every one of these points, a double horizontal step, we obtain a path in $\mathcal{S}_n$. We do not obtain all the paths in $\mathcal{S}_n$, but we obtain all the paths containing a step pair DH or HH, with the appropriate multiplicity (see the two cases marked with (*) in Figure 2.4). On the other hand, if we have a path in $\mathcal{S}_n$ with a step pair DH or HH, by eliminating the second step H, we get a path in $\mathcal{S}_{n-1}$ and therefore we have a bijection between $n-2$ replications of $\mathcal{S}_{n-1}$ and the paths in $\mathcal{S}_n$ having a step pair DH or HH, with the appropriate multiplicity. This shows that the number of step pairs DH or HH is just $(n-2)S_{n-1}$. Finally, if we have double horizontal steps of b different colours, the construction above should be performed by adding b different double horizontal steps, one for each colour. This proves the generalized formula $b(n-2)S_{n-1}$.

∎

Next we show how to find, in an algebraic way, a recurrence relation defining S_n. The method is quite general and is an example of the "indeterminate coefficients" technique applied to a differntial equation. It allows us to determine a recurrence relation defining a set of combinatorial objects as we know the corresponding counting generating function (see [3] for another application of the same technique). The combinatorial interpretation is more complex and is illustrated in the next section.

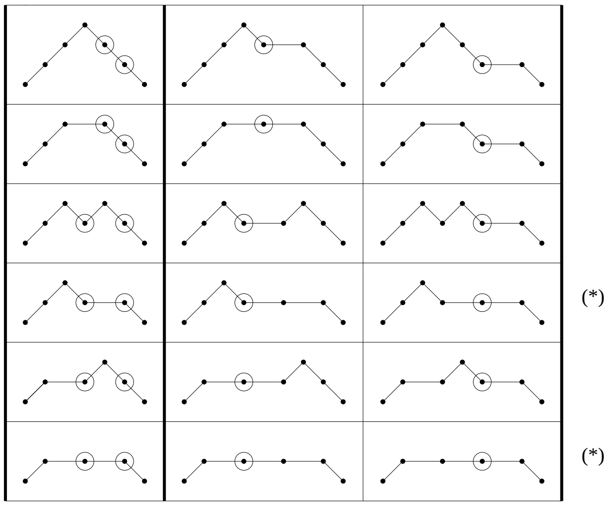

Figure 2.4: From $\mathcal{S}_3$ to the subset of $\mathcal{S}_4$ containing a step pair DH or HH.

Theorem 2.3 *For $n \geq 2$, the following recurrence relation holds:*

$$(b + 2ac)(2n - 1)S_n = (n + 1)S_{n+1} + b^2(n - 2)S_{n-1}$$

with initial conditions $S_1 = ac, S_2 = ac(ac + b)$.

The algebraic proof : We distinguish the following three steps:

S1) The first step consists in finding two rational functions $p_1(t)$ and $p_2(t)$ such that:

$$p_1(t)S'(t) + p_2(t)S(t) = 1.$$

To do so, we set $Q = \sqrt{1 - 2(b + 2ac)t + b^2t^2}$ and obtain:

$$S(t) = \frac{1 - bt - Q}{2}, \qquad S'(t) = \frac{(b + 2ac + b^2t)Q - b(1 - 2t(b + 2ac) + b^2t^2))}{2(1 - 2t(b + 2ac) + b^2t^2)}.$$

S2) Then, we look for $p_1(t)$ and $p_2(t)$ by collecting $p_1(t)S'(t) + p_2(t)S(t)$ with respect to Q and by equating the Q coefficient to 0 and the rest to 1 :

$$(b + 2ac - b^2t)p_1(t) - (1 - 2(b + 2ac)t + b^2t^2)p_2(t) = 0$$

$$b(1-2(b+2ac)t+b^2t^2)p_1(t)-(1-(3b+4ac)t+b(3b+4ac-bt)t^2)p_2(t) = -2(1-2(b+2ac)t+b^2t^2).$$

S3) By solving the previous linear system we find:

$$p_1(t) = \frac{1 - 2(b + 2ac)t + b^2t^2}{ac(1 + bt)}, \qquad p_2(t) = \frac{b + 2ac - b^2t}{ac(1 + bt)}.$$

This solution corresponds to the following differential equation:

$$(1 - 2(b + 2ac)t + b^2t^2)S'(t) + (b + 2ac - b^2t)S(t) = ac(1 + bt),$$

and by extracting the n^{th} coefficient we get, after some symplifying, the proof of the theorem.

The combinatorial proof: See next section. ■

From Theorems 2.1 and 2.3 the following Corollary follows:

Corollary 2.4 *Narayana polynomials satisfy the following recurrence relation:*

$$(2n-1)(1+w)N_{n-1}(w) = (n+1)N_n(w) + (n-2)(1-w)^2 N_{n-2}(w)$$

with initial conditions $N_0(w) = 1$, $N_1(w) = w$.

Proof: The recurrence is easily proved for $w = \frac{b+ac}{ac}$ by substituting S_n with $(ac)^n N_{n-1}((b+ac)/ac)$ in the recurrence for S_n; $a, b, c \in \mathbf{N}$, i.e. the relation holds true for an infinite number of points. Since it is an equality between polynomials, it should be valid for all $w \in \mathbf{R}$. ■

Remark: for a complete combinatorial proof see Sulanke [7].

3 The combinatorial interpretation

In this section we give a combinatorial interpretation of the recurrence

$$(b+2ac)(2n-1)S_n = (n+1)S_{n+1} + b^2(n-2)S_{n-1}; \tag{3.4}$$

our proof is mainly based on the approach of Sulanke [5], who studies the case corresponding to $a = b = c = 1$. The same case has also been studied in [2], where Foata and Zeilberger give a combinatorial interpretation of the recurrence by using a completely different approach.

In order to prove (3.4) we proceed by performing the following actions:

A1) in corrispondence with the term $(b+2ac)(2n-1)S_n$, we consider paths in $\mathcal{S}_n$, take $2n-1$ replications of each path by marking, with a symbol x or y (see below), each of its $2n-1$ non-terminal $D^{[c]}$ steps, and finally take $b+2ac$ equal copies of each path;

A2) in corrispondence with the term $(n+1)S_{n+1}$, we consider paths in $\mathcal{S}_{n+1}$ and take $n+1$ replications of each path by marking, with a symbol z, each of its $n+1$ $U^{[a]}$ and $H^{[b]}$ steps;

A3) in corrispondence with the term $b^2(n-2)S_{n-1}$, we consider paths in $\mathcal{S}_{n-1}$, take $n-2$ replications of each path by marking, with a symbol z, each of its $n-2$ $H^{[b]}$ and non final $D^{[c]}$ steps, and finally take b^2 equal copies of each path;

we then define a bijection between the paths defined in A1) and the paths defined in A2), A3).

Regarding action A1), for each path in $\mathcal{S}_n$, we temporarily index its steps with the integers 1 through $2n-1$ so that each $U^{[a]}$ step and each nonfinal $D^{[c]}$ step receives one integer and each $H^{[b]}$ step receives two consecutive integers. Then we mark each path by selecting an integer from $\{1, \ldots, 2n-1\}$ and marking the corresponding step:

- by the symbol x if the step is $U^{[a]}$ or if the step is $H^{[b]}$ with an odd index,
- by the symbol y if the step is $D^{[c]}$ or if the step is $H^{[b]}$ with an even index.

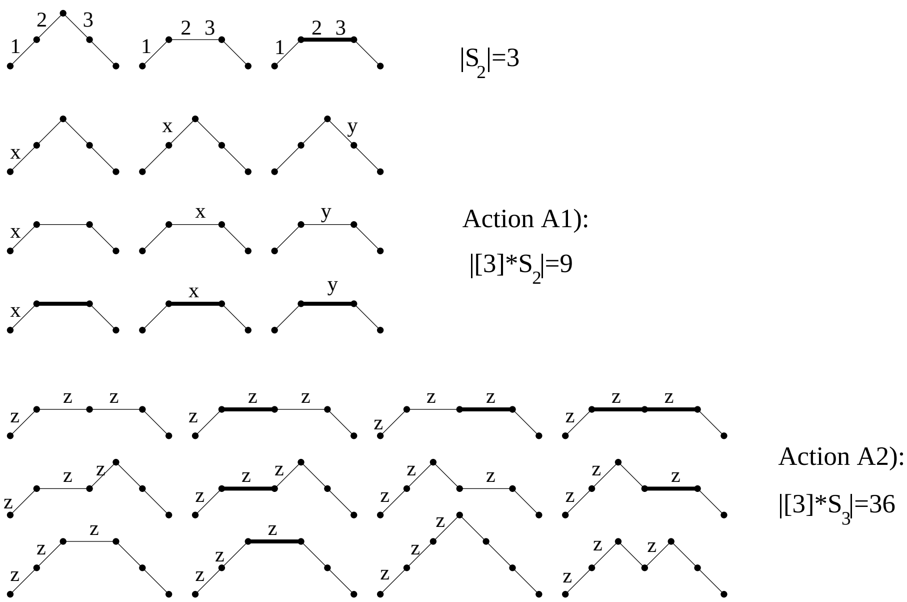

Figure 3.5: Sets $\mathcal{S}_n$, $[2n-1]\times\mathcal{S}_n$ and $[n+1]\times\mathcal{S}_{n+1}$ when $n=2$, $a=c=1$ and $b=2$, and their cardinalities.

We denote the set of such replications as $[2n-1]\times\mathcal{S}_n$; then take $b+2ac$ equal copies of $[2n-1]\times\mathcal{S}_n$ and denote this new set as $(b+2ac)[2n-1]\times\mathcal{S}_n$.

For what concerns action A2), we replicate each path in $\mathcal{S}_{n+1}$ by sequentially marking each of its $U^{[a]}$ or $H^{[b]}$ steps by the symbol z (every path in S_{n+1} has a total of $n+1$ steps $U^{[a]}$ and $H^{[b]}$). We denote this replicated set as $[n+1]\times\mathcal{S}_{n+1}$.

Similarly, for action A3), in $\mathcal{S}_{n-1}$ we replicate each path $n-2$ times by sequentially marking one of its $H^{[b]}$ steps or nonfinal $D^{[c]}$ steps by z and denote this set as $[n-2]\times\mathcal{S}_{n-1}$; finally, we denote as $b^2[n-2]\times\mathcal{S}_{n-1}$ the set we obtain by taking b^2 equal copies of $[n-2]\times\mathcal{S}_{n-1}$.

Figure 3.5 illustrate sets $\mathcal{S}_n$, $[2n-1]\times\mathcal{S}_n$, $[n+1]\times\mathcal{S}_{n+1}$ and $[n-2]\times\mathcal{S}_{n-1}$ when $n=2$ and $a=c=1$ and $b=2$, that is, when we have one colour for up and down steps and two colours for the double horizontal steps. The set $[n+1]\times\mathcal{S}_{n+1}$ is represented by marking by z, in all possible ways, the paths in $\mathcal{S}_{n+1}$. Obviously, in this case, $[n-2]\times\mathcal{S}_{n-1}$ is empty.

In order to prove (3.4), we first define $LEV(p_l)$ as the ordinate of the final point of a step p_l; then observe that a replicated path P in $[2n-1]\times\mathcal{S}_n$ can be decomposed as follows:

$$P = p_1\cdots p_i\cdots p_j\cdots p_k\cdots p_m$$

where:

- $1\leq i\leq j<k\leq m$;
- the step p_j is the step marked by x or y;
- the step p_i is the last $U^{[a]}$ step preceding p_{j+1} for which $LEV(p_i)=LEV(p_j)$; when $p_j=U^{[a]x}$, $i=j$;
- the $D^{[c]}$ step p_k is the first step after p_j for which $LEV(p_k)=LEV(p_j)-1$.

We are now in a position to define the desired bijection:

$$\Psi:(b+2ac)[2n-1]\times\mathcal{S}_n\rightarrow[n+1]\times\mathcal{S}_{n+1}\cup b^2[n-2]\times\mathcal{S}_{n-1}$$

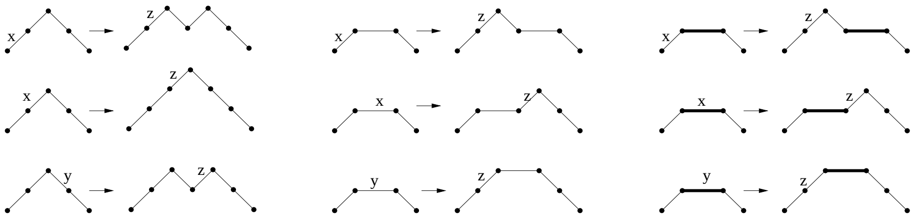

Figure 3.6: Bijection Ψ_1 for $n=2$, $a=c=1$ and $b=2$.

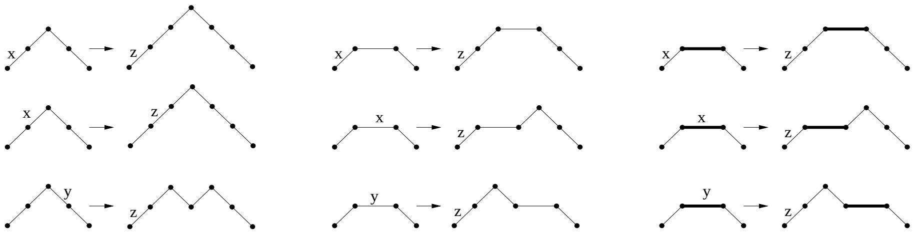

Figure 3.7: Bijection Ψ_2 for $n=2$, $a=c=1$ and $b=2$.

by distinguishing three (pairwise disjoint) bijections Ψ_1, Ψ_2, Ψ_3, such that $\Psi = \Psi_1 \cup \Psi_2 \cup \Psi_3$:

1) Bijection $\Psi_1 : ac[2n-1] \times \mathcal{S}_n \to Q_1$ where $Q_1 \subset [n+1] \times \mathcal{S}_{n+1}$ contains the paths in $[n+1] \times \mathcal{S}_{n+1}$ which contain the step pair $U^{[a]z} D^{[c]}$ or the step triple $U^{[a]z} H^{[b]} D^{[c]}$. We consider a replicated path P in $[2n-1] \times \mathcal{S}_n$:

   1.1) if $p_j = U^{[a]x}, H^{[b]x}$ or $D^{[c]y}$ then

   $$\Psi_1(P) = p_1 \cdots p_i \cdots p_j U^{[a]z} D^{[c]} p_{j+1} \cdots p_k \cdots p_m;$$

   1.2) if $p_j = H^{[b]y}$ then

   $$\Psi_1(P) = p_1 \cdots p_i \cdots p_{j-1} U^{[a]z} H^{[b]} D^{[c]} p_{j+1} \cdots p_m;$$

   In this case, illustrated in Figure 3.6, we essentially add a $U^{[a]}$ and a $D^{[c]}$ step to P obtaining a path in Q_1; since we can do this in ac different ways we transform each path in $ac[2n-1] \times \mathcal{S}_n$ (that is, $[2n-1] \times \mathcal{S}_n$ replicated ac times) into a path in Q_1. Vice versa, every path in Q_1 can be transformed into a path in $ac[2n-1] \times \mathcal{S}_n$ by deleting the pair $U^{[a]z} D^{[c]}$ or by substituting the triple $U^{[a]z} H^{[b]} D^{[c]}$ with the step $H^{[b]}$ and appropriately marking the step p_j.

2) Bijection $\Psi_2 : ac[2n-1] \times \mathcal{S}_n \to Q_2$ where $Q_2 \subset [n+1] \times \mathcal{S}_{n+1}$ (see Observation 3.1 below). We consider a replicated path P in $[2n-1] \times \mathcal{S}_n$:

   2.1) if $p_j = U^{[a]x}, H^{[b]x}$ or $D^{[c]y}$ then

   $$\Psi_2(P) = p_1 \cdots p_i^z \cdots p_j U^{[a]} P' D^{[c]} p_k \cdots p_m,$$

   where $P' = p_{j+1} \cdots p_{k-1}$;

   2.2) if $p_j = H^{[b]y}$ then

   $$\Psi_2(P) = p_1 \cdots p_i^z \cdots p_{j-1} U^{[a]} P' D^{[c]} p_j p_k \cdots p_m,$$

   where $P' = p_{j+1} \cdots p_{k-1}$;

This case is illustrated in Figure 3.7; we essentially add a $U^{[a]}$ and a $D^{[c]}$ step to P obtaining a path in Q_2; since we can do this in ac different ways, we transform each path in $ac[2n-1]\times\mathcal{S}_n$ (that is, $[2n-1]\times\mathcal{S}_n$ replicated ac times) into a path in Q_2. The inverse mapping Ψ_2^{-1} is now defined according to the following observation, which allow us to univocally determine the step p_j :

Observation 3.1 *Let $Q=\Psi_2(P)$ be a path in Q_2. Then it contains the step $U^{[a]z}$ not followed by either the $D^{[c]}$ step or the step pair $H^{[b]}D^{[c]}$ (these cases have already been considered in bijection Ψ_1). Moreover, let q_l be the first (not horizontal) Q's step after $q_i=p_i^z=U^{[a]z}$ such that $LEV(q_l)=LEV(p_i)$. Then the following three cases are possible:*

- $q_l=U^{[a]}$; *this implies* $p_j=p_i$ *in* P;
- $q_lq_{l+1}=D^{[c]}U^{[a]}$; *this implies* $p_j=q_l$ *in* P;
- $q_lq_{l+1}=D^{[c]}H^{[b]}$ *and* $q_{l+2}\neq H^{[b]}$; *this implies* $p_j=q_{l+1}$ *in* P.

Once p_j has been determined, we can transform each path in Q_2 into a path in $ac[2n-1]\times\mathcal{S}_n$.

3) Bijection $\Psi_3: b[2n-1]\times\mathcal{S}_n\rightarrow Q_3\cup Q_4\cup b^2[n-2]\times\mathcal{S}_{n-1}$ (see figure 3.8).

This bijection can be defined by distinguishing three (pairwise disjoint) bijections $\Psi_{3,1}$, $\Psi_{3,2}$, $\Psi_{3,3}$, such that $\Psi_3=\Psi_{3,1}\cup\Psi_{3,2}\cup\Psi_{3,3}$. We consider a replicated path P in $[2n-1]\times\mathcal{S}_n$:

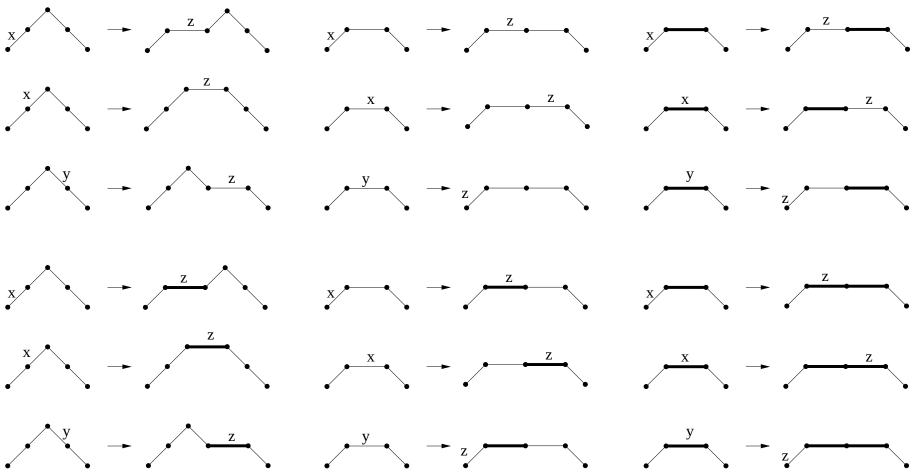

Figure 3.8: Bijection Ψ_3 for $n=2$, $a=c=1$ and $b=2$.

3.1) Bijection $\Psi_{3,1}: T_1\rightarrow Q_3$. Here $T_1\subset b[2n-1]\times\mathcal{S}_n$ (that is, $[2n-1]\times\mathcal{S}_n$ replicated b times) contains paths having $p_j=U^{[a]x}$, $H^{[b]x}$ or $D^{[c]y}$ and $Q_3\subset[n+1]\times\mathcal{S}_{n+1}$ contains paths having the step $H^{[b]z}$:

$$\Psi_{3,1}(P)=p_1\cdots p_i\cdots p_jH^{[b]z}p_{j+1}\cdots p_k\cdots p_m.$$

In this case, we essentially add a $H^{[b]}$ step to P obtaining a path in Q_3; since we can do this in b different ways we transform each path in T_1 into a path in Q_3. Vice versa, every path in Q_3 can be transformed in a path in T_1 by deleting the step $H^{[b]z}$ and by marking the previous step: i) by x, if it is an up or an horizontal step, ii) by y, if it is a down step.

3.2) Bijection $\Psi_{3,2} : T_2 \to Q_4$. Here $T_2 \subset b[2n-1] \times \mathcal{S}_n$ contains paths having $p_{j-1}p_j = U^{[a]}H^{[b]y}$ and $Q_4 \subset [n+1] \times \mathcal{S}_{n+1}$ (see Observation 3.2 below):

$$\Psi_{3,2}(P) = p_1 \cdots p_i^z \cdots p_{j-1} P' H^{[b]} H^{[b]} p_k \cdots p_m,$$

where $P' = p_{j+1} \cdots p_{k-1}$;
In this case we essentially add a $H^{[b]}$ step to P obtaining a path in Q_4; since we can do this in b different ways we transform each path in T_2 into a path of Q_4. The inverse mapping $\Psi_{3,2}^{-1}$ is now defined according to the following observation, which allow us to univocally determine the step p_j :

Observation 3.2 *Let $Q = \Psi_{3,2}(P)$ be a path in Q_4. Then it contains the step $U^{[a]z}$ not followed by either the $D^{[c]}$ step or the step pair $H^{[b]}D^{[c]}$ (these cases have already been considered in bijection Ψ_1). Moreover, let q_l be the first (not horizontal) Q's step after $q_i = p_i^z = U^{[a]z}$ such that $LEV(q_l) = LEV(p_i)$, then $q_l q_{l+1} \dots q_{l+r} = D^{[c]}H^{[b]} \dots H^{[b]}$ with $r > 1$; this implies $p_j = q_{l+r}$ in P.*

Once p_j has been determined, we can transform each path in Q_4 into a path in T_2.

3.3) Bijection $\Psi_{3,3} : T_3 \to b^2[n-2] \times \mathcal{S}_{n-1}$. Here $T_3 \subset b[2n-1] \times \mathcal{S}_n$ contains paths having $p_{j-1}p_j = H^{[b]}H^{[b]y}$ or $p_{j-1}p_j = D^{[c]}H^{[b]y}$:

$$\Psi_{3,3}(P) = p_1 \cdots p_{j-1}^z p_{j+1} \cdots p_m.$$

In this case we essentially eliminate a $H^{[b]}$ step from P in correspondence with step pairs $H^{[b]}H^{[b]y}$ or $D^{[c]}H^{[b]y}$, thus obtaining a path in $[n-2] \times \mathcal{S}_{n-1}$; since there are $b(n-2)S_{n-1}$ such path pairs (see Theorem 2.2) we transform each path in T_3 into a path in $b^2[n-2] \times \mathcal{S}_{n-1}$. The vice versa is obvious.

Finally, we have that Q_1, Q_2, Q_3, Q_4 and T_1, T_2, T_3 are disjoint sets and $Q_1 \cup Q_2 \cup Q_3 \cup Q_4 = [n+1] \times \mathcal{S}_{n+1}$, $T_1 \cup T_2 \cup T_3 = b[2n-1] \times \mathcal{S}_n$.

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