On Some Alternative Characterizations of Riordan Arrays

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Abstract

We give several new characterizations of Riordan Arrays, the most important of which is: if $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is a lower triangular array whose generic element $d_{n,k}$ linearly depends on the elements in a well-defined though large area of the array, then $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is Riordan. We also provide some applications of these characterizations to the lattice path theory.

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1 Introduction

In December 1994, during the second author's visit to the "Dipartimento di Sistemi e Informatica" in Florence (Italy), we began to investigate the enumeration of lattice paths having diagonal steps from the Riordan Array point of view (see, [23, 24]). This problem had been previously studied by Handa and Mohanty [17]; we approached the problem according to the theory discussed in [23, 24].

This theory had previously been developed for lattice paths with "steep"diagonal steps, as illustrated in Figures 1(i) and 1(ii); it is well-known that the arrays determined in this case are Riordan (see [16] for example). But [17] treats lattice paths having "shallow" diagonal steps, illustrated in Figures 1(iii) and 1(iv).

The logical consequence would be to extend the theory of Riordan arrays to the second type of diagonal steps and this is what we want to do. The counting sequences on the main diagonal are obviously the same for both shallow and steep steps if their gradients are reciprocal. This can be verified simply by running the lattice paths backwards (compare Figures 1(i) and 1(iii), or Figures 1(ii) and 1(iv)). It is worth noting, however, that whereas the array in Figure 1(iii) is also a Riordan array, the one in Figure 1(iv) is not.

By using both algebraic and combinatorial techniques, we were able to prove several properties for lattice paths having both kinds of diagonal steps (steep and shallow). To our surprise, we realized that many of these properties were so general that they actually extended the original characterization of Riordan Arrays. The resulting Theorem 2.6 greatly extends the Riordan Array theory, and shows that a lower triangular array $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is Riordan whenever its generic element $d_{n+1,k+1}$ linearly depends on the elements $d_{r,s}$ lying in a well-defined, but large zone of the array (see Figure 2). This is fundamental to the lattice path theory, (see last section), and it is also important in the general Riordan Array theory, because it provides a remarkable characterization of many lower triangular arrays of combinatorial importance, (that is, all the arrays for which a recurrence can be given involving elements belonging to the relevant zone).

These results seem significant to us and led us to divide our work into two parts. In the present paper, we give an account of the new developments in the Riordan Array theory, and use lattice paths as a guiding example. We focus our attention on the new characterizations of

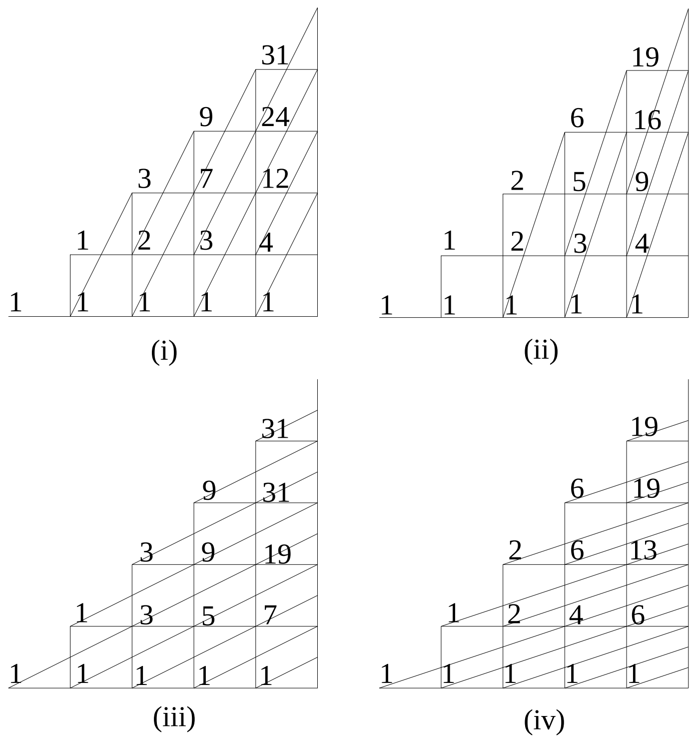


Figure 1: Some arrays illustrating the numeration of lattice paths having diagonal steps.

Riordan Arrays and, in order to maintain the necessary generality, we mainly use an algebraic approach based on generating functions. In our companion paper "Lattice paths with steep and shallow steps" we deal with lattice path problems directly and we use combinatorial proofs to determine which problems correspond to Riordan Arrays and which do not. Even though they are limited to non-negative coefficients, many of these proofs, will constitute the combinatorial counterpart of proofs given in the present paper.

To be more specific, this paper is organized in the following way: in Section 2, we give the definitions and the above-mentioned characterizations of Riordan Arrays. In Section 3, we develop our algebraic theory by giving a number of results concerning the generating functions related to the Riordan Arrays. Finally, in Section 4, we show how the theory can be applied to lattice path problems.

We wish to point out that the combinatorial objects we are mainly interested in are subdiagonal lattice paths in the Cartesian plane. Our paper treats some of the topics studied by Gessel [4] and Labelle [10, 11, 12] but differs from these works in its emphasis on paths not ending on the main diagonal. The simple geometric transformation $(\delta, \delta') \to (\delta + \delta', \delta' - \delta')$ changes underdiagonal paths into paths that never go below the x-axis. This lattice path notation can be called "French notation" because it is mainly used by researchers belonging to the French area (see Goulden and Jackson [7]).

For brevity's, we only outline many of our demonstration and so we refer the reader to the report [15] for the details of the complete proofs.

2 Riordan arrays

By some abuse of language (see Shapiro et al. [23]), a *Riordan array* is a pair $(d(t), h(t))$ in which $d(t)$ and $h(t)$ are analytic functions (or formal power series) such that $d(0) \neq 0$; if $h(0) \neq 0$, then the Riordan array is called *proper*. The pair defines an infinite, lower triangular array $\{d_{n,k}\}_{n,k \in \mathbf{N}}$, in the sense that:

$$d_{n,k} = [t^n]d(t)(th(t))^k,$$

by definition. From this definition, it easily follows that $d(t)(th(t))^k$ is the generating function of column k in the array (in particular, $d(t)$ is the generating function of column 0). The most common example of a Riordan array is the Pascal triangle, in which we have $d(t) = h(t) = 1/(1-t)$. Proper Riordan Arrays are known as "recursive matrices" in the theory of Umbral Calculus (see Barnabei, Brini and Nicoletti [1]). A non-proper Riordan Array $(d(t), h(t))$ can be easily reduced to a proper one: if $h(t)$ has order $s \geq 1$, i.e., $h(t) = t^s v(t)$, with $v(0) \neq 0$, then $(d(t), v(t))$ is a proper Riordan Array and is obtained from $(d(t), h(t))$ by moving every column k up ks positions. The Riordan Array theory allows us to find properties concerning these matrices; for example, we have:

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n]d(t)f(th(t)), \tag{2.1}$$

for every sequence f_k having $f(t)$ as its generating function. A description of the Riordan Array theory together with many examples of it, can be found in Shapiro et al. [23] or in Sprugnoli [24].

Rogers [19] has proved the following, fundamental characterization of proper Riordan Arrays:

Theorem 2.1 *An array $\{d_{n,k}\}_{n,k \in \mathbf{N}}$ is a proper Riordan Array if and only if there exists a sequence $A = \{a_i\}_{i \in \mathbf{N}}$ with $a_0 \neq 0$ such that every element $d_{n+1,k+1}$ (not lying in column 0 or row 0) can be expressed as a linear combination with coefficients in A of the elements in the preceding row, starting from the preceding column on, i.e.:*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots \tag{2.2}$$

Proof: See Rogers [19]. ■

The sum in (2.2) is actually finite because $d_{n,k} = 0$, $\forall k > n$. Sequence A, called the *A-sequence* of the Riordan array, is characteristic in the sense that it determines (and is determined by) function $h(t)$. If $A(t)$ is the generating function of the A-sequence, it can be proven (see Sprugnoli [24]) that $h(t)$ is the solution of the functional equation:

$$h(t) = A(th(t)). \tag{2.3}$$

Conversely, $A(y)$ can be determined by the relation:

$$A(y) = \left[h(t) \,\middle|\, t = yh(t)^{-1}\right],$$

where this notation means that $A(y)$ is obtained by substituting the solution of the functional equation $t = yh(t)^{-1}$ having $t(0) = 0$ for t in $h(t)$. For example, this last relation in the Pascal triangle gives:

$$A(y) = \left[\frac{1}{1-t}\middle| t = y(1-t)\right] = \left[\frac{1}{1-t}\middle| t = \frac{y}{1+y}\right] = 1 + y.$$

Therefore, the A-sequence for the Pascal triangle is $\{1, 1, 0, 0, \ldots\}$ and (2.2) becomes:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1},$$

the well-known basic recurrence for binomial coefficients.

Let us now come to the latest developments in the Riordan Array theory. First of all, as previously mentioned, the A-sequence does not completely characterize a proper Riordan array $(d(t), h(t))$ because the function $d(t)$ is independent of $A(t)$. We therefore prove the following:

Theorem 2.2 *Let $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ be any infinite lower triangular array with $d_{n,n} \neq 0, \forall n \in \mathbf{N}$ (in particular, let it be a proper Riordan array); then a unique sequence $Z = \{z_0, z_1, z_2, \ldots\}$ exists such that every element in column 0 can be expressed as a linear combination of all the elements in the preceding row, i.e.:*

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots \tag{2.4}$$

Proof: Let $z_0 = d_{1,0}/d_{0,0}$. Now we can uniquely determine the value of z_1 by expressing $d_{2,0}$ in terms of the elements in row 1, i.e.

$$d_{2,0} = z_0 d_{1,0} + z_1 d_{1,1} \qquad \text{or} \qquad z_1 = \frac{d_{0,0}d_{2,0} - d_{1,0}^2}{d_{0,0}d_{1,1}}.$$

In the same way, we can determine z_2 by expressing $d_{3,0}$ in terms of the elements in row 2, and by substituting the values just obtained for z_0 and z_1. By proceeding in this way, we determine the Z-sequence in a unique way. ∎

The Z-sequence characterizes column 0, while the A-sequence characterizes all the other columns. The triple $(d_0, Z(t), A(t))$ characterizes a proper Riordan array:

Theorem 2.3 *Let $(d(t), h(t))$ be a proper Riordan array and let $Z(t)$ be the generating function of the array's Z-sequence. Therefore we obtain:*

$$d(t) = \frac{d_0}{1 - tZ(th(t))}.$$

Proof: By the preceding theorem, the Z-sequence exists and is unique. Therefore, equation (2.4) is valid for every $n \in \mathbf{N}$, and we can go on to the generating functions. Since $d(t)(th(t))^k$ is the generating function for column k, we have:

$$\frac{d(t) - d_0}{t} = z_0 d(t) + z_1 d(t)th(t) + z_2 d(t)t^2h(t)^2 + \ldots =$$

$$= d(t)(z_0 + z_1 th(t) + z_2 t^2 h(t)^2 + \ldots) = d(t)Z(th(t)).$$

By solving this equation in $d(t)$, we immediately find the relation desired. ■

The relation can be inverted and this gives us a formula for the Z-sequence:

$$Z(y) = \left[\frac{d(t) - d_0}{td(t)} \middle| t = yh(t)^{-1} \right].$$

The reader can easily apply these formulas to the Pascal triangle, which we can apply the following theorem to:

Theorem 2.4 *Let $d_0 = h_0 \neq 0$. Then $d(t) = h(t)$ if and only if $A(y) = d_0 + yZ(y)$.*

Proof: Let us assume that $A(y) = d_0 + yZ(y)$ or $Z(y) = (A(y) - d_0)/y$. By Theorem 2.3, we have:

$$d(t) = \frac{d_0}{1 - tZ(th(t))} = \frac{d_0}{1 - (tA(th(t)) - d_0 t)/th(t)} = \frac{d_0 th(t)}{d_0 t} = h(t),$$

because $A(th(t)) = h(t)$. Vice versa, by the formula for $Z(y)$, we obtain from the hypothesis $d(t) = h(t)$:

$$d_0 + yZ(y) = \left[d_0 + y \left(\frac{1}{t} - \frac{d_0}{th(t)} \right) \middle| t = yh(t)^{-1} \right] =$$

$$= \left[d_0 + \frac{th(t)}{t} - \frac{d_0 th(t)}{th(t)} \middle| t = yh(t)^{-1} \right] = \left[h(t) \middle| t = yh(t)^{-1} \right] = A(y).$$

■

Riordan arrays having $d(t) = h(t)$ were first introduced by Rogers [19] who called them "renewal arrays". As the concepts of A- and Z-sequences show, what seems essential in a Riordan array is the fact that the elements in a given row linearly depend on the elements of the row above it, starting from the element on the left. It is surprising that this dependence can be made much looser, as the following theorems show (see also the presentation of Shapiro [22]). They greatly increase the applicability range of the Riordan Array theory and play a basic role in our approach to lattice path problems. Let us begin by the following:

Lemma 2.5 *If in a lower triangular array $\{d_{n,k}\}_{n,k \in \mathbf{N}}$ we have:*

$$d_{n+1,k+1} = \sum_{j \geq 0} a_j d_{n,k+j}$$

for some coefficients a_j $(j \geq 0)$, independent of n and k, with $a_0 \neq 0$, then we obtain

$$d_{n,k} = \sum_{j \geq 0} b_j d_{n+1,k+1+j} \tag{2.5}$$

for coefficients b_j $(j \geq 0)$ also independent of n and k. Moreover, if $A(t)$ and $B(t)$ are the generating functions of the two sequences, then $B(t) = A(t)^{-1}$ and therefore:

$$b_0 = \frac{1}{a_0}, \qquad b_n = -\frac{1}{a_0} \sum_{j=1}^{n} b_j a_{n-j} \quad (j \geq 1) \tag{2.6}$$

Proof sketch: By writing formula (2.5) in a matrix form and by using Henrici's result [9, §1.3], we immediately obtain that $A(t)^{-1} = B(t)$ and formula (2.6) is simply the J.C.P. Miller formula for reciprocal formal power series (see Henrici [9, Th. 1.6c]). ■

This lemma is the basis of the following Riordan Array characterizations. We wish to point out that, by Theorem 2.3, the Z-sequence exists for every lower triangular array, and therefore we can implicitly assume its existence in all the subsequent theorems. Our first result is:

Theorem 2.6 *A lower triangular array $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is Riordan if and only if there exists another array $\{\alpha_{i,j}\}_{i,j\in\mathbf{N}}$, with $\alpha_{0,0} \neq 0$, such that every $d_{n+1,k+1}$ $(n, k \geq 0)$ can be expressed as:*

$$d_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq 0} \alpha_{i,j} d_{n-i,k+j} \tag{2.7}$$

Proof sketch: If the array is Riordan, let $\{a_j\}_{j\in\mathbf{N}}$ be its A-sequence: the array defined as $\alpha_{0,j} = a_j$, $\forall j \in \mathbf{N}$, and $\alpha_{i,j} = 0$, $\forall i > 0, j \geq 0$, is exactly as we desired. The proof of the "if" part given in [15] is rather long and complex. It consists in proving that an A-sequence exists for the given array and, therefore, it is Riordan. Lemma 2.5 plays a basic role in this proof. ■

This theorem shows that we can characterize a Riordan Array by means of an A-matrix, rather than by a simple A-sequence. However, while the A-sequence is unique for a given Riordan Array, the A-matrix is not. For example, the following A-matrices, and many others, all define the Pascal triangle (the proof is quite obvious and relies on the basic recurrence for the binomial coefficients):

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We can extend the linear dependence of the generic element $d_{n+1,k+1}$ to allow for elements on its own row, starting from $d_{n+1,k+2}$. In fact, we can prove the following characterization:

Theorem 2.7 *A lower triangular array $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is Riordan if and only if there exist another array $\{\alpha_{i,j}\}_{i,j\in\mathbf{N}}$, with $\alpha_{0,0} \neq 0$, and a sequence $\{\rho_i\}_{i\in\mathbf{N}}$ such that:*

$$d_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq 0} \alpha_{i,j} d_{n-i,k+j} + \sum_{j\geq 0} \rho_j d_{n+1,k+1+j} \tag{2.8}$$

Proof: Here again, the "only if" part is obvious. As to the "if" part, we can eliminate $d_{n+1,k+2}$ from the recurrence, by applying relation (2.8) and by eventually changing the array $\{\alpha_{i,j}\}$ into $\{\alpha'_{i,j}\}$. In the same way, we can subsequently eliminate all the $d_{n+1,k+1+j}$'s. Since only a finite number of them actually appears in the evaluation of $d_{n+1,k+1}$, we can always reduce $d_{n+1,k+1}$ to depend on some array $\{\bar{\alpha}_{i,j}\}_{i=0,1,\ldots}$, which is the left part of a limit array $\{\alpha^*_{i,j}\}$, as happens in Theorem 2.6. Therefore, we can conclude that $\{d_{n,k}\}$ is a Riordan

Array. ■

This result will be used in our study of lattice path problems. Moreover, we can obtain the widest possible characterization of Riordan Arrays (see Theorem 3.7 below):

Theorem 2.8 *A lower triangular array $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is Riordan if and only if there exists another array $\{\alpha_{i,j}\}_{i,j\in\mathbf{N}}$, with $\alpha_{0,0}\neq 0$, and s sequences $\{\rho_j^{[i]}\}_{j\in\mathbf{N}}$ ($i=1,2,\ldots,s$) such that:*

$$d_{n+1,k+1}=\sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j}d_{n-i,k+j}+\sum_{i=1}^{s}\sum_{j\geq 0}\rho_j^{[i]}d_{n+i,k+i+j+1}. \tag{2.9}$$

Proof: Repeated applications of the elimination technique used in the previous theorem's proof. ■

In Figure 2, we try to give a graphic representation of the zones which the generic element $d_{n+1,k+1}$ (denoted by a small disk or "bullet") is allowed to depend on so that the array can be Riordan. The three zones correspond to Theorems 2.6, 2.7 and 2.8, and the only restrictions are that $\alpha_{0,0}\neq 0$ and that the number of rows *below* row n be finite.

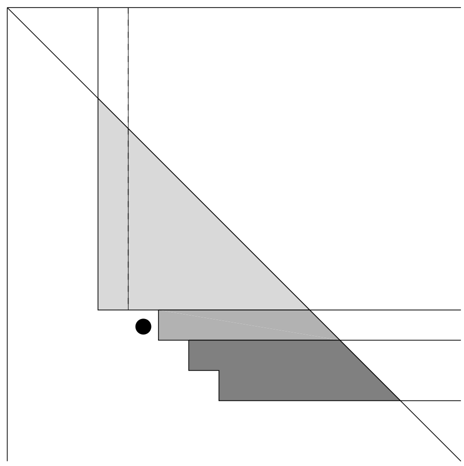


Figure 2: The zones which $d_{n+1,k+1}$ can depend on.

Up to now, we have assumed that $\alpha_{0,0}\neq 0$ because this condition assures that the resulting Riordan Array is proper. However, if we change this hypothesis, but maintain that some $\alpha_{i,0}\neq 0$, for $i>0$, then we obtain a non-proper Riordan Array, i.e., a Riordan Array with $h(0)=0$. This happens under the following conditions:

Theorem 2.9 *Let $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ be an array whose generic element $d_{n+1,k+1}$ is defined by a linear recurrence:*

$$d_{n+1,k+1}=\sum_{i\geq 0}\sum_{j\geq 0}\beta_{i,j}d_{\nu,k+j}\qquad \nu\leq n+S\quad \text{for some}\quad S\in\mathbf{N}.$$

Let γ be the minimum index for which $\beta_{\gamma,0}\neq 0$ and set $\nu'=\nu-\gamma j$. If $\forall\beta_{i,j}\neq 0$ we have $i=n-\nu'$ and, whenever $\nu'>n$, we also have $j\geq\nu'-n$, then $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is a non-proper Riordan Array $(d(t),h(t))$ with $h(t)=t^\gamma v(t)$ and $v(0)\neq 0$.

Proof: The theorem's conditions allow us to define a new array $\{d'_{n,k}\}_{n,k\in\mathbf{N}}$ whose generic element $d'_{n+1,k+1}$ is given by:

$$d'_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j}d'_{n-i,k+j} + \sum_{i\geq 1}^{s}\sum_{j\geq 0}\rho_j^{[i]}d'_{n+i,k+i+j+1},$$

where $\alpha_{i,j} = \beta_{n-\nu',j}$ when $\nu' \leq n$, and $\rho_j^{[i]} = \beta_{\nu'-n,j}$ when $\nu' > n$. The number s exists thanks to the condition $\nu \leq n + S$, for some S. This is actually the definition of a proper Riordan Array, in which $d'_{n,k} = d_{n+k\gamma,k}$ because the columns of $\{d'_{n,k}\}$ are the columns of $\{d_{n,k}\}$ moved γk positions up. If $(d(t), v(t))$ is the new proper Riordan Array, then we should have $h(t) = t^\gamma v(t)$. ■

In Section 4, we will examine an example of a non-proper Riordan Array in connection with a lattice path problem.

3 Generating functions

As previously noted, the A-sequence and the function $h(t)$ of a Riordan Array are strictly related to each other. This fact allows us to think that $h(t)$ can be deduced from the A-matrix $\{\alpha_{i,j}\}_{i,j\in\mathbf{N}}$ and the set of sequences $\{\rho_j^{[i]}\}_{j\in\mathbf{N}}$ for $i = 0, 1, \cdots, s$. Then, after finding the function $h(t)$, we can also find the A-sequence by determining its generating function $A(t)$.

Almost always, $d_{n+1,k+1}$ only depends on the elements of a finite number of rows above it; therefore, instead of treating a global generating function for the A-matrix, let us examine a sequence of generating functions $P^{[0]}(t), P^{[1]}(t), P^{[2]}(t), \ldots$ corresponding to the rows $0, 1, 2, \ldots$ of the A-matrix, i.e.:

$$P^{[0]}(t) = \alpha_{0,0} + \alpha_{0,1}t + \alpha_{0,2}t^2 + \alpha_{0,3}t^3 + \cdots$$

$$P^{[1]}(t) = \alpha_{1,0} + \alpha_{1,1}t + \alpha_{1,2}t^2 + \alpha_{1,3}t^3 + \cdots$$

and so on. Moreover, let $Q^{[i]}(t)$ be the generating function for the sequence $\{\rho_j^{[i]}\}_{j\in\mathbf{N}}$. Thus we have:

Theorem 3.1 *If $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is a Riordan Array whose generic element $d_{n+1,k+1}$ is defined by formula (2.9) through the A-matrix $\{\alpha_{i,j}\}_{i,j\in\mathbf{N}}$ and the set of sequences $\{\rho_j^{[i]}\}_{j\in\mathbf{N}}$, $i = 1, 2, \ldots, s$, then the functions $h(t)$ and $A(t)$ for $\{d_{n,k}\}$ are given by the following implicit expressions:*

$$h(t) = \sum_{i\geq 0} t^i P^{[i]}(th(t)) + \sum_{i=1}^{s} th(t)^{i+1}Q^{[i]}(th(t)). \tag{3.1}$$

$$A(t) = \sum_{i\geq 0} t^i A(t)^{-i} P^{[i]}(t) + t\sum_{i=1}^{s} A(t)^i Q^{[i]}(t). \tag{3.2}$$

Proof: Let $d_k(t) = d(t)(th(t))^k$ be the generating function of column k of the Riordan Array; from (2.9) we deduce:

$$\frac{d_{k+1}(t)}{t} = \sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j}t^i d_{k+j}(t) + \sum_{i=1}^{s}\sum_{j\geq 0}\rho_j^{[i]}t^{-i}d_{k+i+j+1}(t)$$

$$\frac{d(t)(th(t))^{k+1}}{t} = \sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j}t^i d(t)(th(t))^{k+j} + \sum_{i=1}^{s}\sum_{j\geq 0}\rho_j^{[i]}t^{-i}d(t)(th(t))^{k+i+j+1}.$$

We can now divide everything by $d(t)(th(t))^k$:

$$h(t) = \sum_{i\geq 0}t^i\sum_{j\geq 0}\alpha_{i,j}(th(t))^j + \sum_{i=1}^{s}t^{-i}(th(t))^{i+1}\sum_{j\geq 0}\rho_j^{[i]}(th(t))^j.$$

We now go on to the generating functions $P^{[i]}(t)$ and $Q^{[i]}(t)$ and formula (3.1) immediately follows. Finally, by applying formula (2.3) we obtain the expression (3.2) for $A(t)$. ■

As stated in the proof of Theorem 2.6, this theorem allows us to give some explicit formulas for the element a_n of the A-sequence. By extracting the coefficient of t^n, we find:

$$a_n = [t^n]A(t) = \sum_{i\geq 0}[t^{n-i}]B(t)^i P^{[i]}(t) + \sum_{i=1}^{s}[t^{n-1}]A(t)^i Q^{[i]}(t)$$

$$a_n = \sum_{i=0}^{n}\sum_{j=0}^{n-i}b_j^{(i)}\alpha_{i,n-i-j} + \sum_{i=1}^{s}\sum_{j=0}^{n-i}a_j^{(i)}\rho_{n-i-j}^{[i]},$$

where $a_j^{(i)}$ and $b_j^{(i)}$ denote the coefficients of t^j in the formal power series $A(t)^i$ and $B(t)^i = A(t)^{-i}$, respectively. As far as Theorem 2.6 is concerned, we have $s = 0$ and so:

$$a_n = \sum_{i=0}^{n}\sum_{j=0}^{n-i}b_j^{(i)}\alpha_{i,n-i-j},$$

which agrees with the values $a_0 = b_0^{(0)}\alpha_{0,0}$ and $a_1 = b_0^{(0)}\alpha_{0,1} + b_0^{(1)}\alpha_{1,0} = \alpha_{0,1} + b_0\alpha_{1,0} = \alpha_{0,1} + \alpha_{1,0}/\alpha_{0,0}$ (see the proof of Theorem 2.6 in [15]). As to Theorem 2.7, we have:

$$a_n = \sum_{i=0}^{n}\sum_{j=0}^{n-i}b_j^{(i)}\alpha_{i,n-i-j} + \sum_{j=0}^{n-i}a_j\rho_{n-i-j},$$

which only depends on the previously computed a_j values .

The generic element $d_{n+1,k+1}$ often only depends on the two previous rows and sometimes on the elements of its own row. In this case, the functional equation (3.2) reduces to a second degree equation in $A(t)$ and, as a result, we can give an explicit expression for the generating function of the A-sequence.

Theorem 3.2 *Let $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ be a Riordan Array whose generic element $d_{n+1,k+1}$ only depends on the two previous rows and, in case, on its own row. If $P(t), \bar{P}(t)$ and $Q(t)$ are the* *generating functions for the coefficients of this dependence, i.e., $P(t) = P^{[0]}(t)$, $\bar{P}(t) = P^{[1]}(t)$ and $Q(t) = Q^{[1]}(t)$, then we have:*

$$A(t) = \frac{P(t) + \sqrt{P(t)^2 + 4t\bar{P}(t)(1 - tQ(t))}}{2(1 - tQ(t))}. \tag{3.3}$$

Proof sketch: Formula (3.2) gives two solutions for $A(t)$ and the one having $A(0) = 0$ must be discarded because we always assume that $a_0 \neq 0$. ■

It is worth noting that if $Q(t) = 0$, that is $d_{n+1,k+1}$ does not depend on the elements of its own row, then we have:

$$A(t) = \frac{P(t) + \sqrt{P(t)^2 + 4t\bar{P}(t)}}{2},$$

which is quite useful in several cases. When the dependence is more complicated, it is naturally more difficult to give an explicit expression for the A-sequence.

As shown in the previous section, $h(t)$ is related to $A(t)$ and $d(t)$ is related to $Z(t)$, the Z-sequence generating function. Since the Z-sequence exists for every lower triangular array (see Theorem 2.2), every recurrence defining $d_{n+1,0}$ in terms of the other elements in the array can be accepted as a good definition of column 0. Therefore, in analogy to (2.9), let us assume that we have the following linear relation:

$$d_{n+1,0} = \sum_{i\geq 0}\sum_{j\geq 0} \zeta_{i,j} d_{n-i,j} + \sum_{i=1}^{s}\sum_{j\geq 0} \sigma_j^{[i]} d_{n+i,i+j}. \tag{3.4}$$

In general, there is no connection between the $\zeta_{i,j}$'s and the $\alpha_{i,j}$'s or between the $\rho_j^{[i]}$'s and the $\sigma_j^{[i]}$'s and so we take the following generating functions into account:

$$R^{[0]}(t) = \zeta_{0,0} + \zeta_{0,1}t + \zeta_{0,2}t^2 + \zeta_{0,3}t^3 + \cdots$$

$$R^{[1]}(t) = \zeta_{1,0} + \zeta_{1,1}t + \zeta_{1,2}t^2 + \zeta_{1,3}t^3 + \cdots$$

(etc.) and $S^{[i]}(t) = \sum_{j\geq 0} \sigma_j^{[i]} t^j$. The coefficients defining $d_{n+1,k+1}$ and $d_{n+1,0}$ are sometimes the same ones, in the sense that:

$$\zeta_{i,j} = \alpha_{i,j+1} \quad \text{and} \quad \sigma_j^{[i]} = \rho_j^{[i]} \qquad \forall i, \forall j.$$

In this case, we say that *column 0 is unprivileged* and we obtain the following formulas for our generating functions:

$$R^{[i]}(t) = \frac{P^{[i]}(t) - \alpha_{i,0}}{t} \quad \text{and} \quad S^{[i]}(t) = Q^{[i]}(t)$$

for every i which $R^{[i]}$, $P^{[i]}$, $S^{[i]}$ and $Q^{[i]}$ are well-defined for.

At any rate, we can easily prove the following:

Theorem 3.3 *If $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is a Riordan Array whose elements in column 0 are defined by a relation (3.4), then the function $d(t)$ is given by the following formula:*

$$d(t) = \frac{d_{0,0}}{1 - \sum_{i\geq 0} t^{i+1}R^{[i]}(th(t)) - t\sum_{i=1}^{s} h(t)^i S^{[i]}(th(t))}. \tag{3.5}$$

Proof sketch: We go on to generating functions and find (3.5) by solving in $d(t)$. ■

When column 0 is unprivileged, the formula for $d(t)$ can be drastically simplified and, nevertheless, actually covers a large class of lattice path problems. For this reason, we state it as a separate theorem:

Theorem 3.4 *If $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is a Riordan Array whose column 0 is unprivileged, then $d(t)$ is given by the formula:*

$$d(t) = \frac{d_{0,0}h(t)}{\sum_{i\geq 0}\alpha_{i,0}t^i}. \tag{3.6}$$

Proof: We simply take the denominator in formula (3.5) and substitute $R^{[i]}(t)$ and $S^{[i]}(t)$ by their counterparts when column 0 is unprivileged; we then use Theorem 3.1's first result. ■

Besides being important for its own sake, this theorem also allows us to prove a very interesting characterization of "renewal arrays", i.e., Riordan Arrays having $d(t) = h(t)$, when column 0 is unprivileged:

Corollary 3.5 *Let $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ be a Riordan Array whose column 0 is unprivileged; then $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is a renewal array if and only if the following two conditions are satisfied: i) $d_{n+1,k+1}$ only depends on $d_{n,k}$ and not on any other element in column k; ii) $\alpha_{0,0} = d_{0,0}$.*

Proof: If column 0 is unprivileged and $d(t) = h(t)$, then by (3.6) we have: $\sum_{i\geq 0}\alpha_{i,0}t^i = d_{0,0}$; therefore $\alpha_{i,0} = 0, \forall i \geq 1$ and this is equivalent to condition i). Only $\alpha_{0,0} = d_{0,0}$ is left and constitutes condition ii). Vice versa, if column 0 is unprivileged, then condition i) implies: $\sum_{i\geq 0}\alpha_{i,0}t^i = \alpha_{0,0}$, so $d(t) = d_{0,0}h(t)/\alpha_{0,0}$, and so condition ii) gives $d(t) = h(t)$. ■

We wish to conclude this section by introducing an important result concerning the characterizations proven in the previous section. By means of generating functions, we can show that Theorem 2.8 gives the largest possible characterization of Riordan Arrays. In other words, we can show that if $d_{n+1,k+1}$ depends on elements not contained in the grey zones of Figure 2, then $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is not a Riordan Array. It is worth noting that if $d_{n+1,k+1}$ depends on some elements $d_{\nu,\kappa}$ with $\nu > n$ and $\kappa < k+1+\nu-n$, then the recurrence is not well-defined, and the computation of $d_{n+1,k+1}$ enters an infinite loop and its indexes keep growing, and, as a result, $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is actually not defined. We must therefore show that $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is not a Riordan Array when $d_{n+1,k+1}$ depends on some element $d_{\nu,\kappa}$ with $\nu \leq n$ and $\kappa < k$. The following theorem shows this under the same conditions as Theorem 2.6 (no $\rho_j^{[i]}$ is involved) and with $\kappa \geq k-1$. Actually, this is sufficient for our purposes because the presence of some $\rho_j^{[i]}$'s does not change the proof. Moreover, the method is virtually the same when $\kappa < k-1$ (only a few technical aspects are slightly modified).

Theorem 3.6 *If the generic element $d_{n+1,k+1}$ in an array $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is defined by the recurrence:*

$$d_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq -1} \alpha^*_{i,j} d_{n-i,k+j} \qquad (d_{n,-1} = 0, \forall n \in \mathbf{N})$$

with some $\alpha^*_{i,-1} \neq 0$, then $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is not a Riordan Array.

Proof sketch: By assuming that the array is Riordan and by going on to generating functions, we obtain the contradiction that all the $\alpha^*_{i,-1}$ are zero. This proves the theorem. ∎

4 Lattice path problems

In the foregoing sections, we assumed that Figure 1 can provide a representation of four sample lattice path enumeration problems on the integer square lattice. To state it in more formal—though less abstract—terms, a lattice path of m steps is a finite sequence $(s_1, \cdots, s_m)$ of ordered pairs $s_i = ((x_{i-1}, y_{i-1}), (x_i, y_i))$, $1 \leq i \leq m$, of lattice points such that:

a) $x_0 = y_0 = 0$;

b) for $1 \leq i \leq m$, $x_i = x_{i-1} + \delta_i, y_i = y_{i-1} + \delta'_i$;

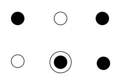
c) the pairs $(\delta_i, \delta'_i), 1 \leq i \leq m$, are drawn from a set of permissible step templates; and

d) these permissible step templates obey some conditions on their occurrence.

We say that such a path starts at the origin $(0,0)$ and ends at (x_m, y_m).

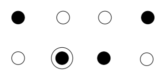
Therefore, in all the examples illustrated in Figure 1, we refer to the step templates $(0,1), (1,0)$, and (δ, δ'), subject to the condition that $0 \leq y_i \leq x_i$, for $0 \leq i \leq m$, and only the choice of (δ, δ') is at issue. In the examples that include Figure 1(i) and 1(ii), $\delta = 1$, while δ' is a positive integer, so the gradient δ'/δ of the step template is large and therefore the step is said to be "steep". In the examples that include Figure 1(iii) and 1(iv), $\delta' = 1$, while δ is a positive integer, and we get a small gradient. Therefore, the step is said to be "shallow". We could obviously give some more complicated examples that allow combinations of these step templates, and sometimes may have different colours. There is vast literature on lattice path enumeration, and we particularly want to mention the following: [2, 3, 5, 6, 7, 8, 13, 14, 16, 17, 18, 20, 21]. In all the examples illustrated in Figure 1, we obtain lower triangular arrays $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ where $d_{n,k}$ is the number of paths which start at $(0,0)$ and end at $(n, n-k)$, as illustrated in Figure 3. To be more precise, we are going to examine some lattice paths having templates in the class $T = \{(\delta, \delta') | \delta, \delta' \in \mathbf{N}, \delta + \delta' > 0\} \cup \{(\delta, \delta') | \delta < 0, \delta' > 0\}$. We denote a step template (δ, δ') having $\delta \geq 0$ by $e^\delta n^{\delta'}$, where e stands for *east* and n for *north*; a template is *steep* if $\delta \leq \delta'$ and is *shallow* if $\delta > \delta' + 1$; if $\delta = \delta' + 1$ the template will be called *almost steep*. A step template (δ, δ') having $\delta < 0$ will be denoted by $w^{|\delta|} n^{\delta'}$, where w stands for *west*; for convenience's sake, we consider every template of this kind as being *steep* too. In Figure 4(a) we illustrate the different kinds of templates and distinguish the sets of steep from almost steep templates by

n/k	0	1	2	3	4
0	1				
1	1	1			
2	3	2	1		
3	9	7	3	1	
4	31	24	12	4	1



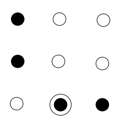
(i)

n/k	0	1	2	3	4
0	1				
1	1	1			
2	2	2	1		
3	6	5	3	1	
4	19	16	9	4	1



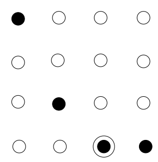
(ii)

n/k	0	1	2	3	4
0	1				
1	1	1			
2	3	3	1		
3	9	9	5	1	
4	31	31	19	7	1



(iii)

n/k	0	1	2	3	4
0	1				
1	1	1			
2	2	2	1		
3	6	6	4	1	
4	19	19	13	6	1



(iv)

Figure 3: The lower triangular arrays resulting from Figure 1.

two different shades of grey. These templates play a fundamental role in our approach to the lattice path theory.

We can now define a lattice path problem R as a pair (R_A, R_Δ), where:

- R_A is a possibly infinite set of templates in T;
- R_Δ is a possibly infinite set of steep templates in T.

An R-path is a path composed of steps with templates in R, and satisfies the following conditions: i) if a step ends on the main diagonal $x - y = 0$, then its template should belong to R_Δ; otherwise ii) the template should belong to R_A. There is an important definition related to these conditions: let R_S be the subset of R_A made up of all its steep templates; if $R_S \neq R_\Delta$, then we say that R is a lattice path problem *with privileged access* to the main diagonal; otherwise, if $R_S = R_\Delta$, then R *have unprivileged access* to the main diagonal. None of the examples in Figure 1 have privileged access to the main diagonal; an example having privileged access will be given further on.

Thanks to these definitions, we can now prove our main result regarding lattice path problems. When we go from a lattice path problem $R = (R_A, R_\Delta)$ to the lower triangular array counting the paths from the origin to the point $(n, n-k)$, as we did to go from Figure 1 to Figure 3, we simply change the two sets R_A and R_Δ into two recurrences: one valid in general, the other only valid for the column corresponding to its main diagonal, i.e., for column 0. It is immediately clear that a template (δ, δ') translates into the dependence of

$d_{n+1,k+1}$ from $d_{n-\delta+1,k+\delta'+1-\delta}$. Since we always have $d_{0,0} = 1$, corresponding to the empty path, these recurrences completely define the array. It is worth noting that a problem with privileged (unprivileged) access to the main diagonal is translated into an array with privileged (unprivileged) column 0.

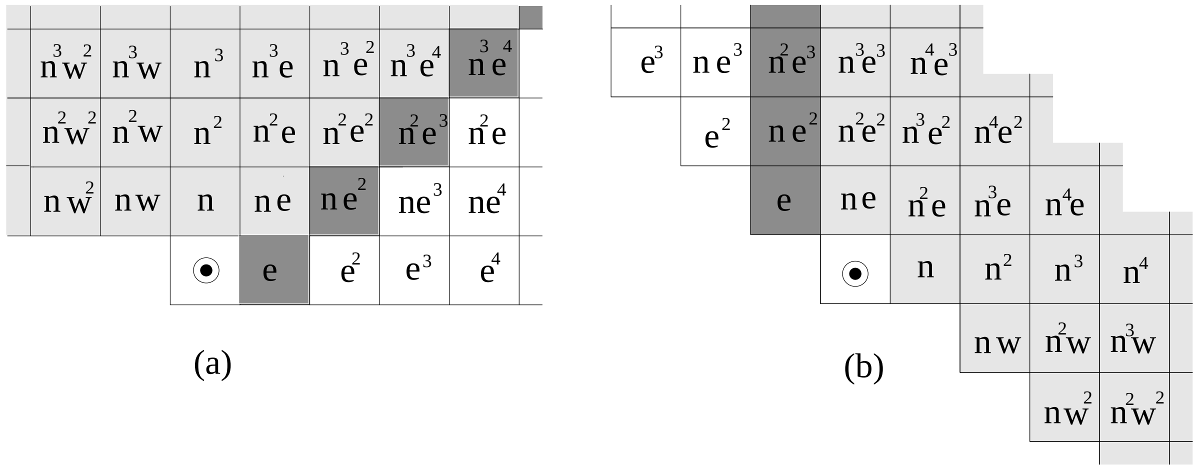


Figure 4: Possible steps originating from a given point in $\mathbf{Z}^2$ and their positions in the corresponding triangular array: e=east, n=north, w=west.

In Figure 4(b) we show, in terms of step templates, the dependence of the generic element $d_{n+1,k+1}$ (or $d_{n+1,0}$) (denoted by "$\odot$") from other elements in the array. Since R_Δ is only made up of steep templates, the recurrence for $d_{n+1,0}$ does not depend on any elements in the white or dark-grey zones, and this makes very good sense. All these considerations help us to prove our main theorem:

Theorem 4.1 *Let (R_A, R_Δ) be a lattice path problem and let $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ be its corresponding counting array. Then $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is a Riordan Array if and only if R_A is made of both steep templates and at least one almost steep template, and a number S exists such that for every $(\delta,\delta') \in R_A \cup R_\Delta$ with $\delta < 0$, we have $\delta' < S$. Besides, $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is proper if R_A contains the almost steep template $(1,0)$.*

Proof: This is an obvious consequence of Theorems 2.8 and 3.7; the condition on S implies that there is only a finite number of rows below row n which $d_{n+1,k+1}$ (or $d_{n+1,0}$) may depend on. ■

This theorem justifies our initial statement that only case (iv) in Figure 1 does not correspond to a Riordan Array. The Riordan Array theory can be applied to the other cases to solve the lattice path problems, as we are now going to show:

In Figure 3(i), we give a schematic illustration of the dependence of $d_{n+1,k+1}$ from the other elements in the array and obtain the recurrence:

$$d_{n+1,k+1} = d_{n,k} + d_{n,k+2} + d_{n+1,k+2}.$$

However, we can directly use Theorem 3.1 to obtain the function $h(t)$ because $P^{[0]}(t) = 1+t^2$ and $Q^{[1]}(t) = 1$, and therefore $h(t)$ is the solution to the equation:

$$h(t) = 1 + t^2 h(t)^2 + t h(t)^2.$$

Since $\alpha_{0,0} = 1$, the Riordan Array is proper, i.e., $h(0) \neq 0$; this implies that:

$$h(t) = \frac{1 - \sqrt{1 - 4t - 4t^2}}{2t(1+t)} = 1 + t + 3t^2 + 9t^3 + 31t^4 + 113t^5 + \cdots$$

The conditions in Corollary 3.6 are now satisfied and so the Riordan Array is actually a renewal array and $d(t) = h(t)$. Finally, the A-sequence can be computed with formula (3.3), where $\bar{P}(t) = 0$ and we find the simple expression:

$$A(t) = \frac{1+t^2}{1-t} = 1 + t + 2t^2 + 2t^3 + 2t^4 + 2t^5 + \cdots$$

The Riordan Array theory can now be used to obtain some information about these paths. For example, the total number N_n of paths extending up to $x = n$ is given by the row sums, which can be computed by means of formula (2.1) with $f(t) = (1-t)^{-1}$:

$$\mathcal{N}(t) = \sum_{n \geq 0} N_n t^n = \frac{d(t)}{1 - t h(t)} = \frac{1 - 2t - \sqrt{1 - 4t - 4t^2}}{4t^2}.$$

We can obtain the average height of these paths in a similar way. We begin by computing the weighted row sums:

$$\mathcal{W}(t) = \sum_{n \geq 0} W_n t^n = \frac{t d(t) h(t)}{(1 - t h(t))^2} = \frac{1 - 4t - (1-2t)\sqrt{1 - 4t - 4t^2}}{8t^3};$$

we then extract the asymptotic value for W_n and N_n, by means of Darboux' method:

$$N_n \approx \frac{\sqrt{4 - 2\sqrt{2}}}{4} \frac{(2 + 2\sqrt{2})^{n+2}}{(2n+3)\sqrt{\pi(n+2)}}, \qquad W_n \approx \frac{(2 - \sqrt{2})\sqrt{4 - 2\sqrt{2}}}{8} \frac{(2 + 2\sqrt{2})^{n+3}}{(2n+5)\sqrt{\pi(n+3)}}.$$

Finally, the quantity desired is computed by subtracting the value of W_n/N_n from n because the weight of an element measures the distance from the diagonal along the y-axis.

For the problem illustrated in Figure 1(ii), we have $P^{[0]}(t) = 1 + t^3$ and $Q^{[1]}(t) = 1$; therefore, $h(t)$ is given by the solution of the third degree equation:

$$h(t) = 1 + t^3 h(t)^3 + t h(t)^2. \tag{4.1}$$

By Corollary 3.6, this is a renewal array and $d(t) = h(t)$. By using the Lagrange Inversion Formula (see Goulden and Jackson [7]), we can find an explicit expression (although not a closed formula) for the generic element $d_{n,k}$. If we multiply (4.1) by t, and set $y = t h(t)$ so that $y(0) = 0$, then we have $y = t(1 + y^3)/(1 - y)$ and, therefore:

$$d_{n,k} = [t^n] d(t) (t h(t))^k = [t^n] \frac{y}{t} y^k = [t^{n+1}] y^{k+1} =$$

$$= \frac{k+1}{n+1}[y^{n-k}]\left(\frac{1+y^3}{1-y}\right)^{n+1} = \frac{k+1}{n+1}\sum_{j=0}^{n-k}\binom{n+1}{j/3}\binom{2n-k-j}{n-k-j},$$

which can be easily checked against the true values given in Figure 3(ii) (if $j/3$ is not an integer, the binomial coefficient should be taken as 0).

Finally, for the problem illustrated in Figure 1(iii), we have $P^{[0]}(t) = P^{[1]}(t) = Q^{[1]}(t) = 1$; therefore, $h(t)$ is given by the solution of $h(t) = 1 + t + th(t)^2$; that is:

$$h(t) = \frac{1 - \sqrt{1 - 4t - 4t^2}}{2t}.$$

By Corollary 3.6, this is not a renewal array and we should compute $d(t)$ by means of formula (3.5) in Theorem 3.4:

$$d(t) = \frac{1}{1 - th(t)} = \frac{1 - \sqrt{1 - 4t - 4t^2}}{2t(1+t)}$$

which, as announced, is the same as for the problem illustrated in Figure 1(i).

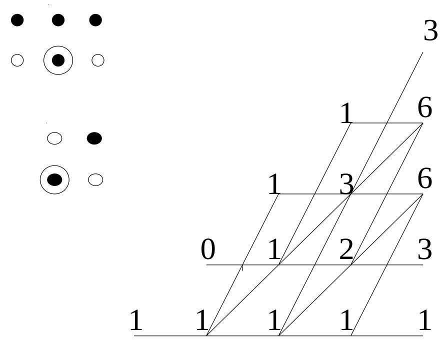
The problem in Figure 1(iv) does not correspond to any Riordan Array; we do not try to solve it here and invite the reader to refer to our paper "Lattice paths with steep and shallow steps".

We want to conclude this section with some other examples that illustrate various ways of applying the results obtained in the previous sections.

The first example is $R = (R_A, R_\Delta)$ with $R_A = \{(1,0),(1,1),(1,2)\}$ and $R_\Delta = \{(1,2)\}$. Since $R_S = \{(1,1),(1,2)\} \neq R_\Delta$, we have a problem involving privileged access to the main diagonal. In this case, we know the A- and Z-sequences, for which we have $A(t) = 1 + t + t^2$ and $Z(t) = t$. By formula (2.3) and Theorem 2.3, we find:

$$d(t) = \frac{1 + t - \sqrt{1 - 2t - 3t^2}}{2t(1+t)}, \qquad h(t) = \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t^2}.$$

The resulting triangle is shown in Figure 5; its row sums are:



n\k	0	1	2	3	4	5
0	1					
1	0	1				
2	1	1	1			
3	1	3	2	1		
4	3	6	6	3	1	
5	6	15	15	10	4	1

Figure 5: Walks with e, ne and n^2e steps having privileged access to the main diagonal.

$$\sum_{k=0}^{n} d_{n,k} = [t^n]\frac{d(t)}{1 - th(t)} = [t^n]\frac{1}{\sqrt{1 - 2t - 3t^2}}$$

which are the well-known trinomial coefficients.

Another example, is $R = (R_A, R_\Delta)$ $R_A = \{(1,k)|k \in \mathbf{N}\} \cup \{(0,1)\}$ and $R_\Delta = \{(1,k)|k \in \mathbf{N}\}$, i.e., having unprivileged access to the main diagonal. In this case, even though we have an infinite number of step templates, we can easily find $P^{[0]}(t) = 1/(1-t)$ and $Q^{[1]}(t) = 1$. Figure 6 illustrates the situation corresponding to this problem; by Theorem 3.1, we find that $h(t)$ is the solution of the following third-degree equation:

$$h(t) = \frac{1}{1-th(t)} + th(t)^2, \qquad \text{or} \qquad h(t) = \frac{1}{(1-th(t))^2}.$$

If we set $y = th(t)$, so that $y(0) = 0$, the previous relation becomes:

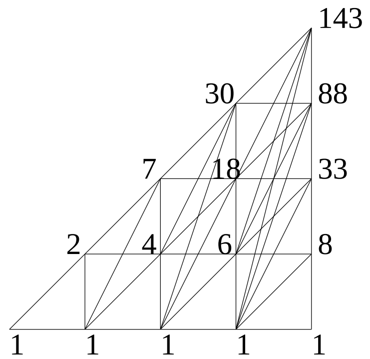
$$y = \frac{t}{(1-y)^2},$$

and we are now able to apply the Lagrange Inversion Formula. By Corollary 3.6, this is a renewal array and we have:

$$d_{n,k} = [t^n]d(t)(th(t))^k = [t^n]\frac{1}{t}(th(t))^{k+1} = [t^{n+1}]y^{k+1} =$$

$$= \frac{1}{n+1}[y^n]\frac{(k+1)y^k}{(1-y)^{2n+2}} = \frac{k+1}{n+1}[y^{n-k}](1-y)^{-2n-2} = \frac{k+1}{n+1}\binom{3n-k+1}{n-k}.$$

Another example having unprivileged access to the main diagonal and with an infinite



n\k	0	1	2	3	4
0	1				
1	2	1			
2	7	4	1		
3	30	18	6	1	
4	143	88	33	8	1

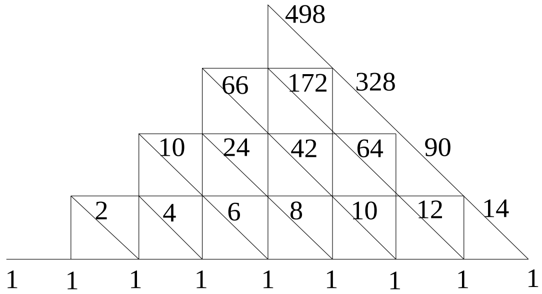
Figure 6: Walks with n and $n^k e$ steps, $k \in \mathbf{N}$, and their corresponding array.

number of step templates is $R_A = \{(\delta, \delta')|\delta \in \mathbf{N}, \delta' = \delta + 1\}$. We now have $P^{[i]}(t) = 1$, $\forall i \geq 0$ and can therefore find:

$$h(t) = \frac{1}{1-t}.$$

By Corollary 3.6, this is not a renewal array, but formula (3.6) in Theorem 3.5 gives $d(t) = h(t)/\sum_{i\geq 0} t^i = 1$. Therefore, the Riordan Array is $D = (1, (1-t)^{-1})$ and $d_{n,k} = \binom{n-1}{k-1}$.

Let us now consider an example having some north-west steps; more precisely, let $R_A = \{(1,0), (0,1), (-1,1)\}$ with unprivileged access to the main diagonal. In Figure 7, we show the first values corresponding to this problem. If we want to compute the first n rows of the resulting array, we must begin by computing the first $2n$ starting values on the x-axis. We then go on to compute the values on the line $y = 1$, and so forth, reducing the number of values computed by one each time. This corresponds to evaluating a sufficient number of values on the diagonal $n = k$ in the resulting Riordan Array. We then examine one diagonal at a time and reduce the number of its rows by one. In this problem we have



n\k	0	1	2	3	4
0	1				
1	2	1			
2	10	4	1		
3	66	24	6	1	
4	498	172	42	8	1

Figure 7: A problem with an *nw* template.

$P^{[0]}(t) = Q^{[1]}(t) = Q^{[2]}(t) = 1$, and formula (3.1) gives $h(t)$ as a solution of the third-degree equation:

$$h(t) = 1 + th(t)^2 + th(t)^3. \tag{4.2}$$

By Corollary 3.6, this is a renewal array and the Lagrange Inversion Formula can be used to find an explicit expression for $d_{n,k}$. By setting $y = h(t) - 1$ so that $y(0) = 0$, formula (4.2) becomes $y = t(1+y)(2+y)$ and we therefore have for $n \neq k$:

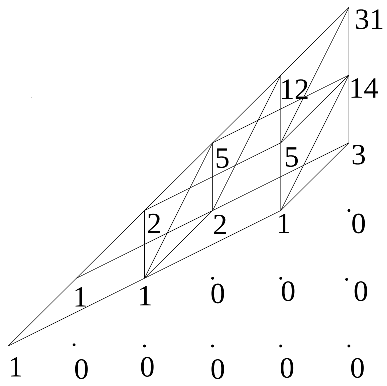
$$[t^n]d(t)(th(t))^k = [t^{n-k}](1+y)^{k+1} = \frac{k+1}{n-k}[y^{n-k-1}](1+y)^{2n-k}(2+y)^{n-k} =$$

$$= \frac{k+1}{n-k}\sum_{j=0}^{n-k-1}\binom{n-k}{j+1}\binom{2n-k}{j}2^{j+1},$$

which can be checked against the values shown in Figure 7.

We conclude by studying a problem corresponding to a non-proper Riordan Array. Let $R_A = \{(0,1),(1,1),(2,1),(1,2)\}$ with unprivileged access to the main diagonal. In Figure 8, we illustrate the problem schematically. We follow Theorem 2.9 and modify the templates in order to obtain a problem relative to a proper Riordan Array. Each template (δ, δ') becomes a template $(\bar{\delta}, \bar{\delta}')$, where $\bar{\delta} = \delta + \gamma(\delta' - \delta)$ and $\bar{\delta}' = \delta' + \gamma(\delta' - \delta)$. In our case, $\gamma = 1$ and so the new templates are $\bar{R}_A = \{(1,2),(1,1),(1,0),(2,3)\}$. For this problem, we have $P^{[0]}(t) = 1 + t + t^2$ and $P^{[1]}(t) = t^2$. This gives the relation $h(t) = 1 + th(t) + t^2h(t)^2 + t^3h(t)^2$; that is:

$$h(t) = \frac{1 - t - \sqrt{1 - 2t - 3t^2 - 4t^3}}{2t^2(1+t)}.$$



n\k	0	1	2
0	1		
1	1		
2	2	1	
3	5	2	
4	12	5	1
5	31	14	3

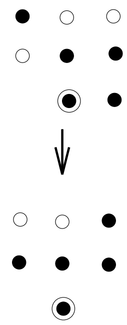


Figure 8: A problem corresponding to a non-proper Riordan Array.

Since we have $d(t) = h(t)$ by Corollary 3.6, we can conclude that the original problem corresponds to the non-proper Riordan Array:

$$D = \left(\frac{1 - t - \sqrt{1 - 2t - 3t^2 - 4t^3}}{2t^2(1+t)}, \frac{1 - t - \sqrt{1 - 2t - 3t^2 - 4t^3}}{2t(1+t)} \right).$$

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