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A generating tree approach to schedule printing devices

Donatella Merlini

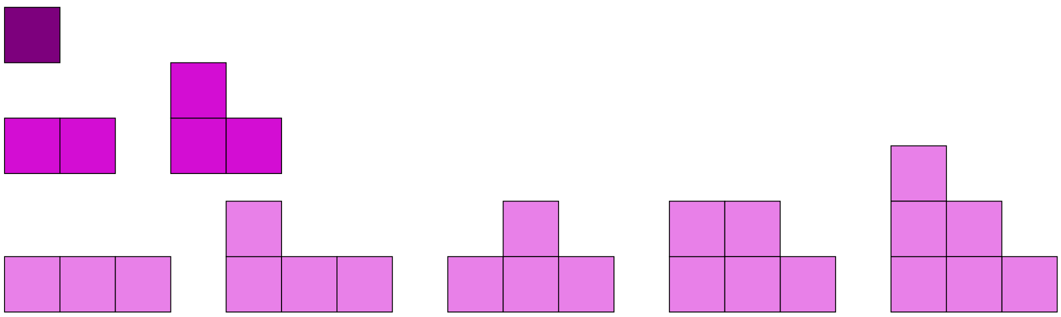
Dipartimento di Sistemi e Informatica
via Lombroso 6/17, 50134
University of Florence, Italy

Join work with R. Sprugnoli and M. C. Verri

$p^{[r]}$-histograms

For $p \geq 1$ and $r = 1 \ldots p$, a $p^{[r]}$-*histogram* of length n is a histogram whose last column only contains r cells and, whenever a column is composed by k cells, then the next column contains at least $k - p$ cells.

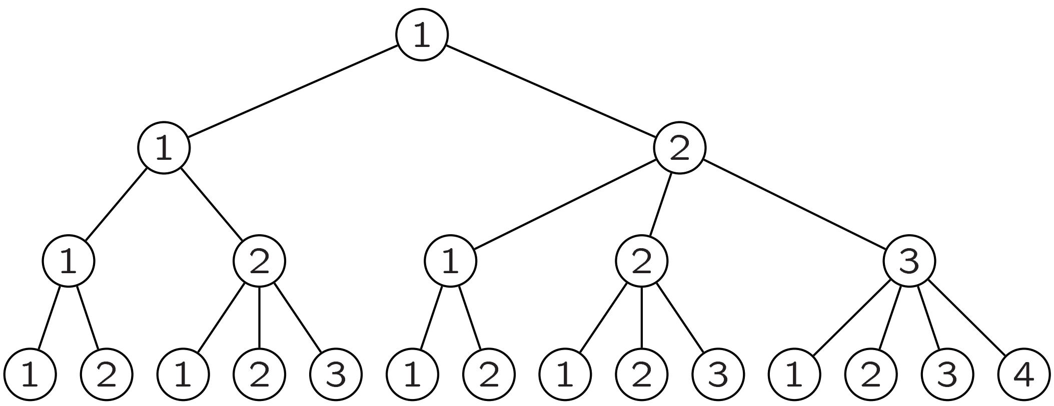
The $1^{[1]}$-histograms with at most 3 columns.



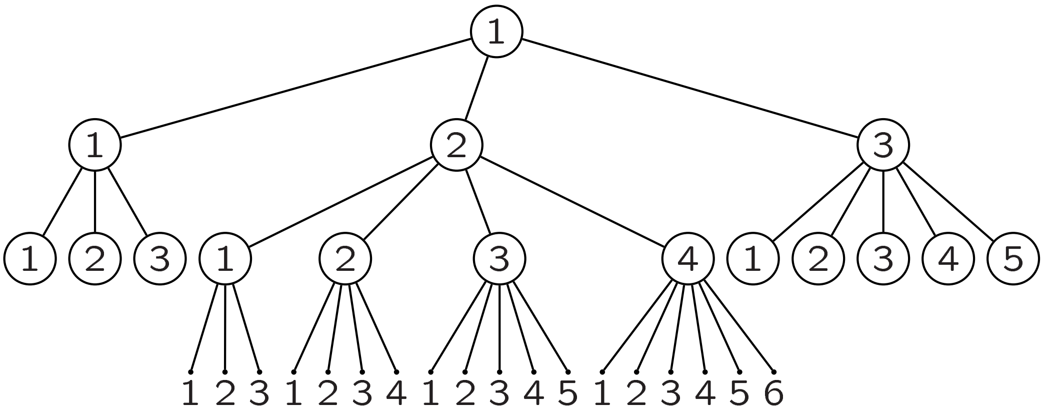
A $p^{[r]}$-histogram corresponds to a path in the generating tree produced by the following specification:

$$\begin{cases} root: & (r) \\ rule: & (k) \quad \rightarrow (1) \cdots (k+p-1)(k+p) \end{cases}$$

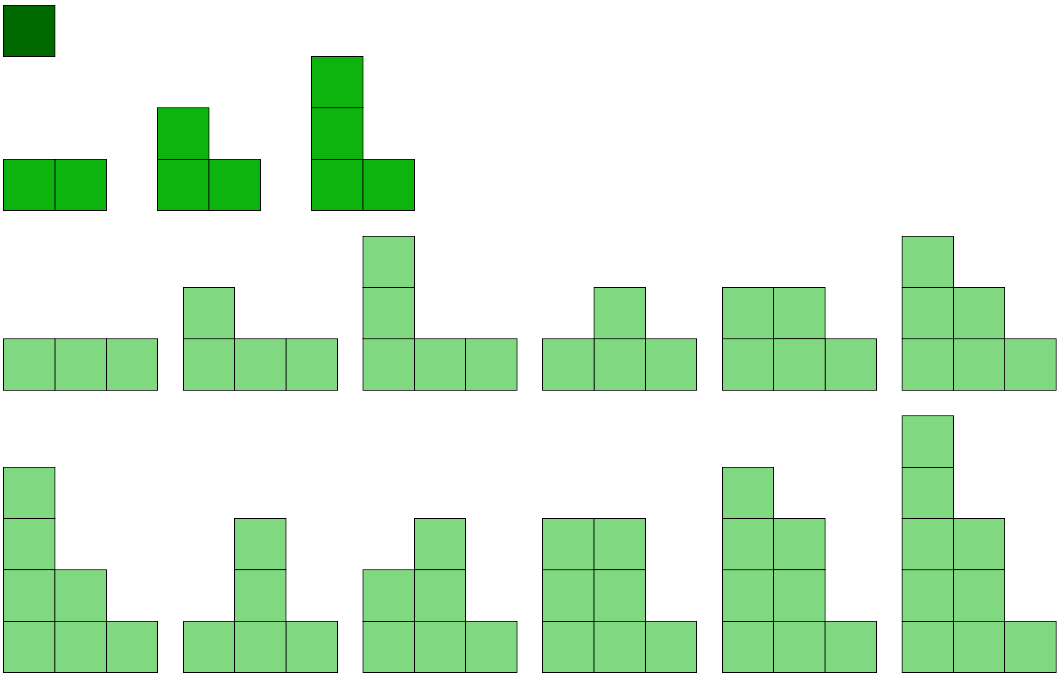
The partial generating tree for $p = 1, r = 1$.



The partial generating tree for $p = 2, r = 1$.



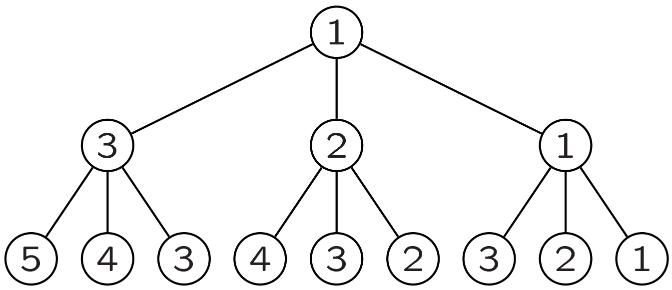
The $2^{[1]}$-histograms with at most 3 columns



Counting $p^{[1]}$-histograms

There exists a $1-1$ correspondence between the set of e-labelled $(p+1)$-ary trees with n nodes and paths in the generating tree with $r=1$.

A complete e-labelled ternary tree ($p=2$)



There exists a $1-1$ correspondence between the $p^{[1]}$-histograms with n columns and the paths in the generating tree with $r=1$ and hence with $(p+1)$-ary trees with n nodes.

$p^{[1]}$-histograms with n-columns are counted by the generating function $T(t)$ counting the number T_n of $(p+1)$-ary trees with n nodes:

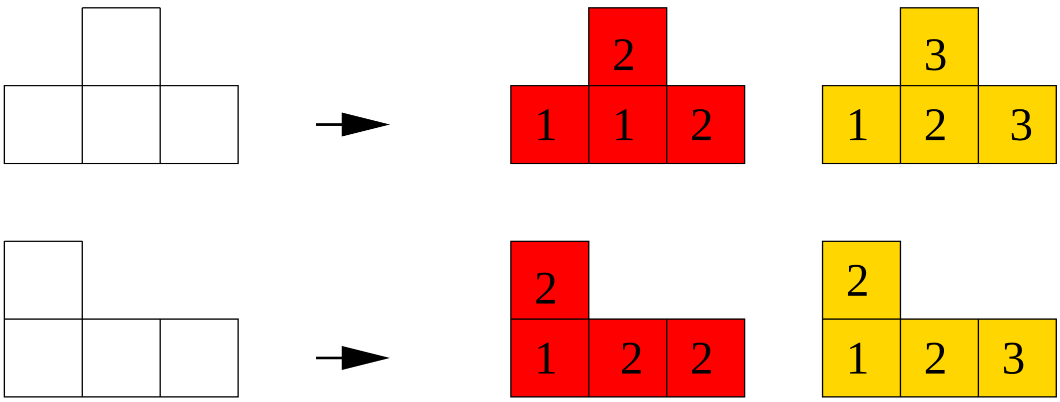
$$T(t) = 1 + tT(t)^{p+1}$$

$$T_n = [t^n]T(t) = \frac{1}{pn+1}\binom{(p+1)n}{n}$$

In general, $p^{[r]}$-histograms of length n, $r = 1, \ldots, p$ are counted by:

$$\mathcal{T}_n = [t^n]\sum_{k=1}^{p} T(t)^{p+k} = [t^n]\frac{T(t)^p - 1}{t} = \frac{p}{n}\binom{p(n+1)+n-1}{n-1}$$

Given a particular $p^{[r]}$-histogram it can correspond to different utilizations of the p printers from different numbers of users; we fill the histogram with numbers that indicate in which order the users' requests are treated.



- every increase of h in the histogram corresponds to the arrival of at least h and at most $h+p$ new users, depending on the number of users which exit at the same time

- every decrease of h (we have $h \leq p$) corresponds to the departure of h users

- users exit in incresing order (user # 1, then user # 2, and so on)

The model

Given p printers which operate for at most n times slots, we consider the set of $p^{[r]}$-histograms of length n, for all $r = 1, \ldots, p$, each taken by considering all possible scheduling. For each printer, a time slot corresponds to printing a single page and at each time slot at least one printer operates. We consider a uniform distribution on all scheduling.

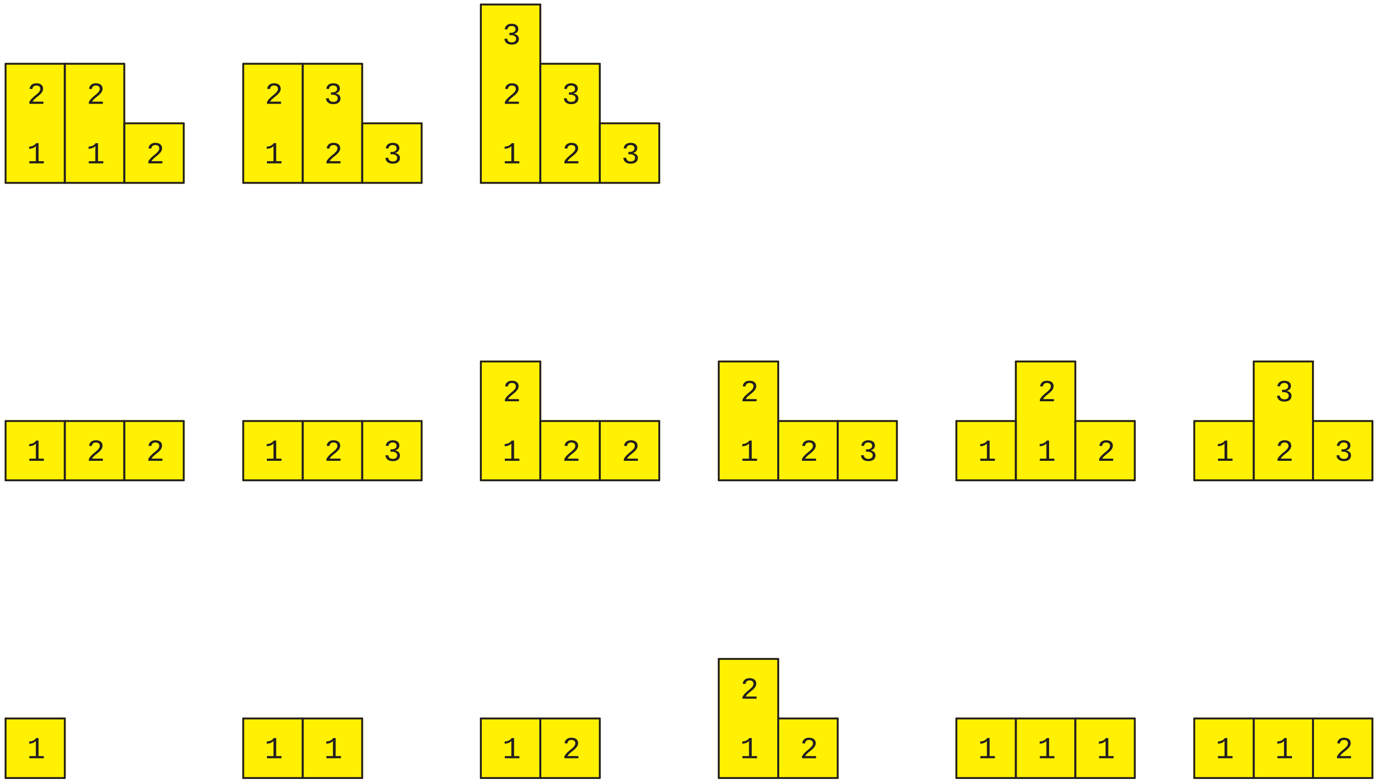
We are interested in studying the average number of users, $AveU_n$ the average waiting time of an user, $AveW_n$, and the average length of the queue, $AveQ_n$, among all the possible configurations of length n.

To do so, we need to know the number of configurations, the total number of users and the total area among all the possible configurations of lenght n, $r = 1, \ldots, p$.

$$AveU_n = \frac{(total\ \#\ of\ users)_n}{(\#\ of\ configurations)_n}$$

$$AveW_n = \frac{(total\ area)_n}{p \times (total\ \#\ of\ users)_n}$$

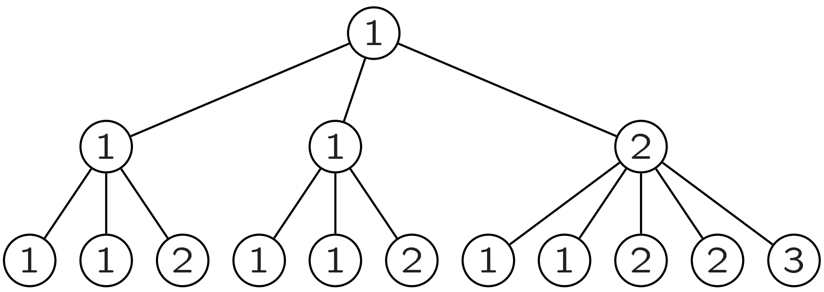
$$AveQ_n = \frac{(total\ area)_n}{p \times n \times (\#\ of\ configurations)_n}$$


2 2
1 1 2
2 3
1 2 3
3
2 3
1 2 3
1 2 2
1 2 3
2
1 2 2
2
1 2 3
2
1 1 2
3
1 2 3
1
1 1
1 2
2
1 2
1 1 1
1 1 2

In the case $p = 1$ we have some results:

$$\begin{cases} root: & (1) \\ rule: & (k) \quad \rightarrow (1)(1)\cdots(k)(k)(k+1) \end{cases}$$

The partial generating tree for all schedulings



$$C(t) = \frac{1 - 3t - \sqrt{1 - 6t + t^2}}{4t} =$$

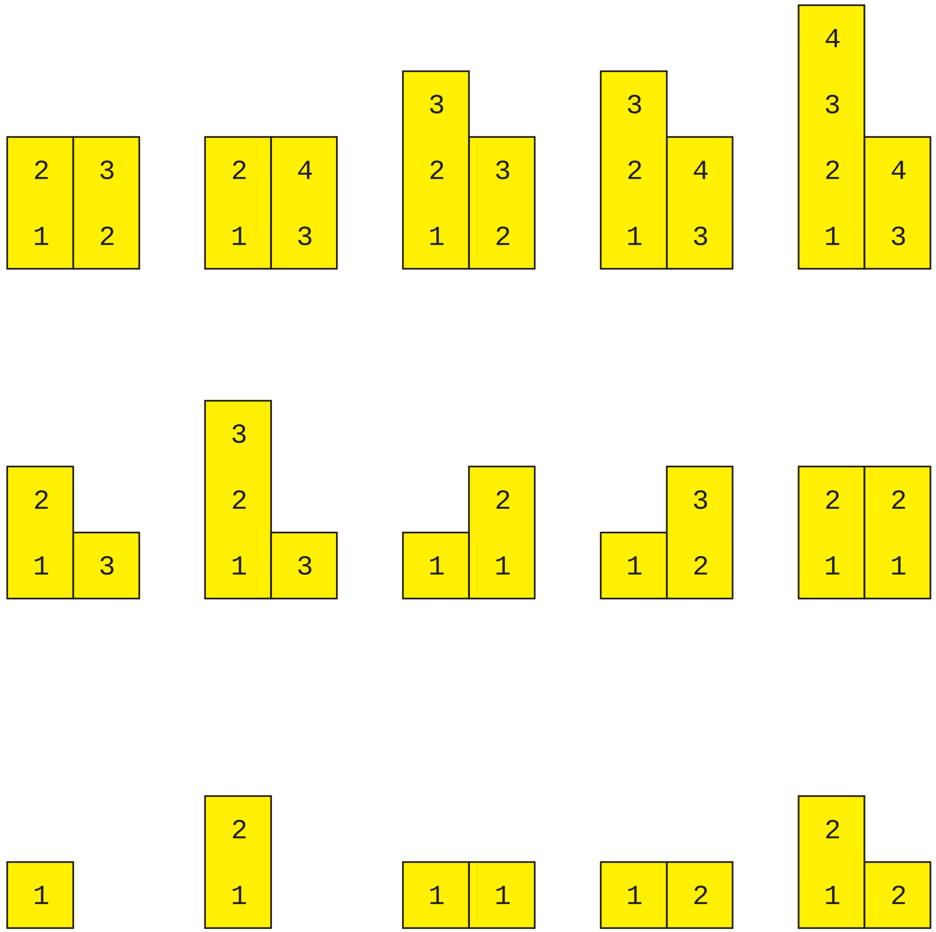
$$= t + 3t^2 + 11t^3 + 45t^4 + 197t^5 + 903t^6 + O(t^7)$$

$$C_n = \sum_{k \geq 1} N_{n,k} 2^{k-1}$$

$$U(t) = t + 5t^2 + 26t^3 + 138t^4 + 743t^5 + 4043t^6 + O(t^7)$$

$$U_n = \sum_k N_{n,k} \sum_{j=0}^{n-k} \binom{n-k}{j} (k+j)$$

$$A(t) = t + 7t^2 + 44t^3 + 268t^4 + 1609t^5 + 9583t^6 + O(t^7)$$


2 3
1 2
2 4
1 3
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2 4
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2 4
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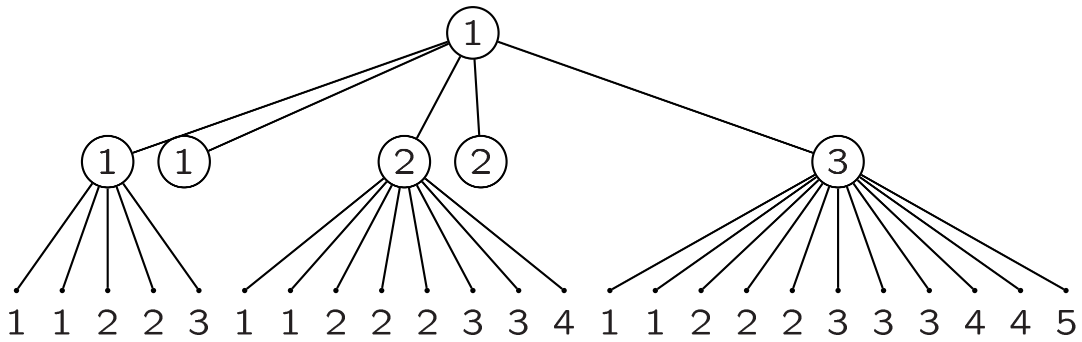
1 3
3
2
1 3
2
1 1
3
1 2
2 2
1 1
1
2
1
1 1
1 2
2
1 2

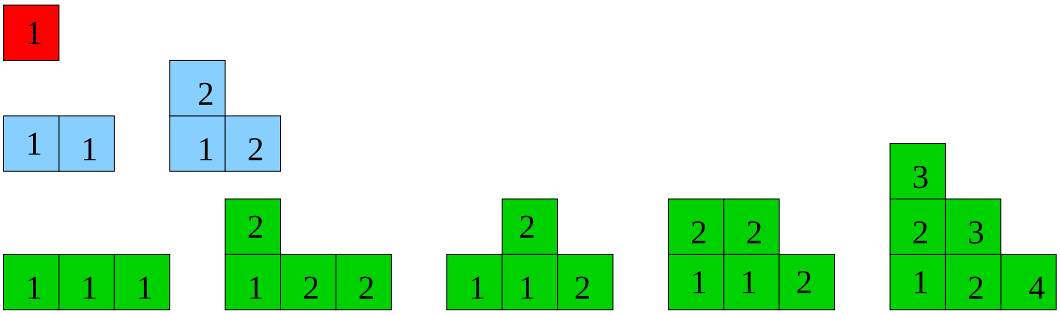
The case $p = 2$ is more complex:

$$\left\{ \begin{array}{lll} root: & (r) & \quad r = 1, 2 \\ rule: & (1) & \to (1)^2(2)^2(3) \\ & (k) & \to (1)^2(2)^3 \cdots (k)^3(k+1)^2(k+2) \end{array} \right.$$

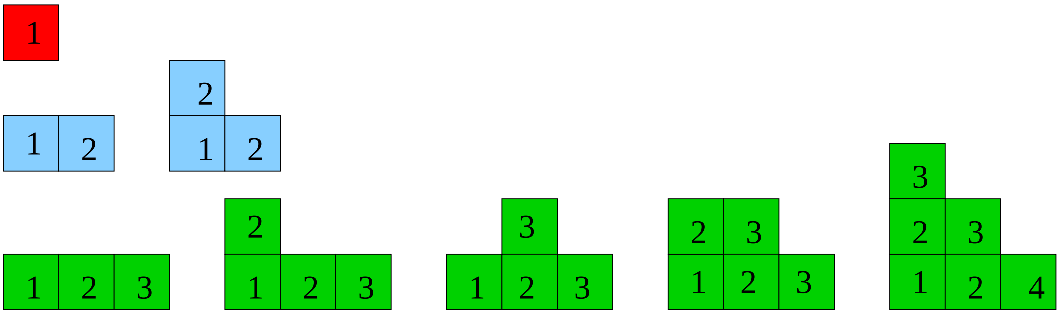
The partial generating tree for all schedulings ($r = 1$)



MIN CASE



MAX CASE



$$AveU_n^{MIN} \leq AveU_n \leq AveU_n^{MAX}$$

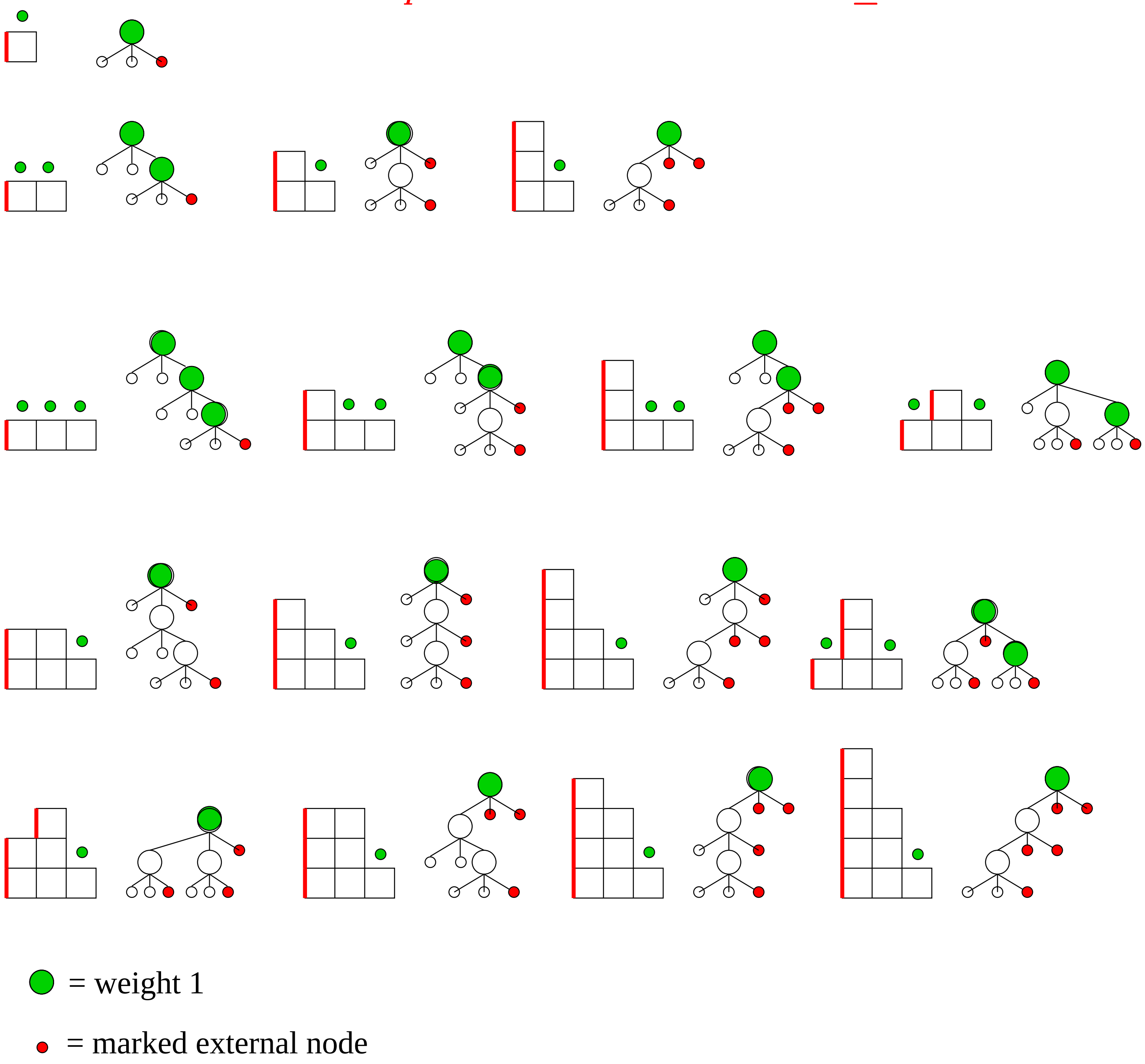
$$AveW_n^{MAX} \leq AveW_n \leq AveU_n^{MIN}$$

$$AveQ_n \leq AveQ_n^{MIN=MAX}$$

Here we consider a uniform distribution on histograms (at each histogram corresponds a unique schduling, in both cases). The number of configurations are counted by $\mathcal{T}_n$.

In order to compute the total number of users in both MIN and MAX cases and the total area we first study the configurations corresponding to $p^{[1]}$-histograms.

The case $p = 2$ and $r = 1$ with $n \leq 3$.



Total # of users among $p^{[1]}$-histograms: MIN CASE

Theorem: The number $T_{n,k}$ of $p^{[1]}$-histograms of length n corresponding to MIN number of users $= k$ (k is also the total number of rises) is equal to the number of $(p+1)$-ary trees with n internal nodes and k marked external nodes.

$$T(t,w) = \sum T_{n,k} t^n w^k =$$

$$= 1 + t \left((T(t,w) - 1) \sum_{i=2}^{p} (T(t,w) - 1 + w)^i \right.$$

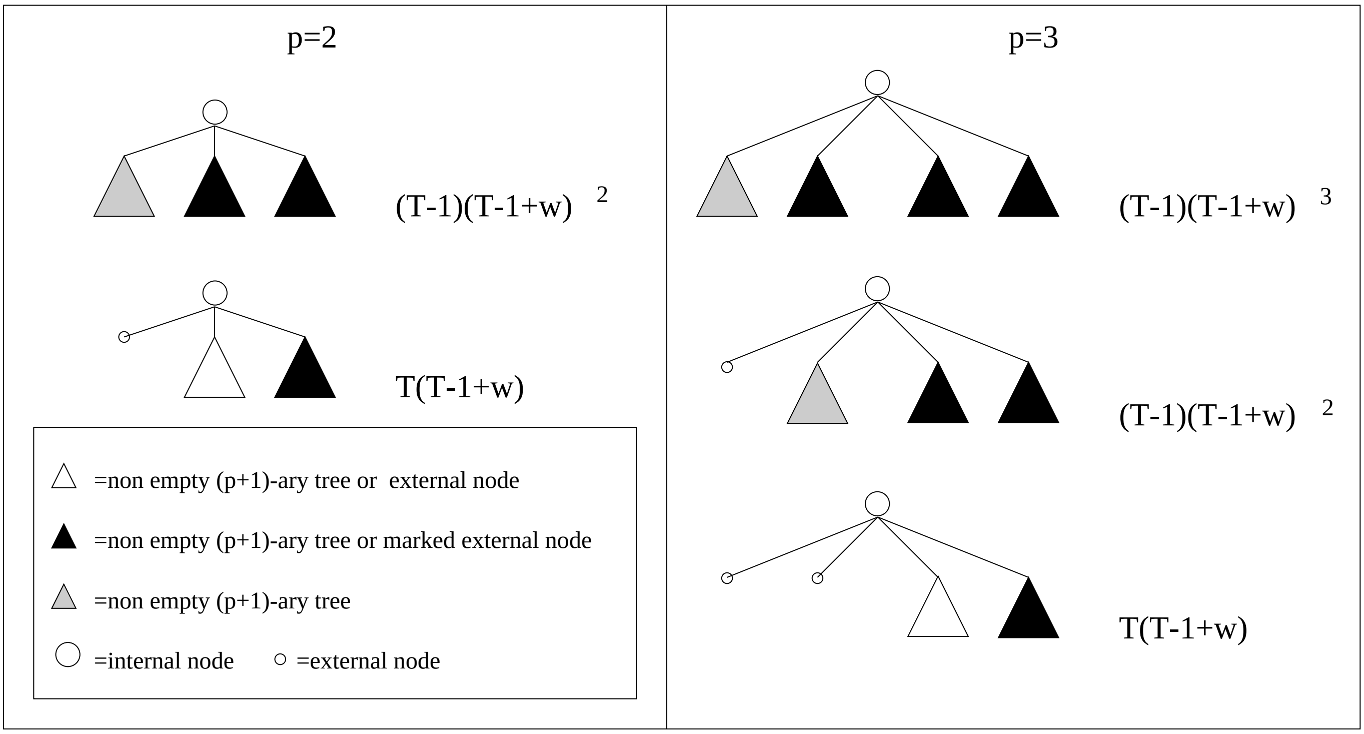
$$\left. + T(t,w)(T(t,w) - 1 + w) \right).$$

$$U_n^{MIN} = [t^n] \left. \frac{\partial T(t,w)}{\partial w} \right|_{w=1} =$$

$$= [t^n] - \frac{t+p}{p} + \frac{t}{T(t) - 1} - \frac{t(p+1)^2}{p(pT(t) - p - 1)} =$$

$$= \frac{(p(n-1)+1)(p(n-1)+n)}{n} T_{n-1}$$

Counting marked external nodes for $(p+1)$-ary tree: $T = T(t, w)$.



Total # of users among $p^{[1]}$-histograms: MAX CASE

Theorem: The number $i_{n,k}$ of $p^{[1]}$-histograms of lenght n having a total idle time of lenght k is equivalent to the number of $(p+1)-ary$ tree with n internal nodes and weight k. Hence:

$$U_n^{MAX} = pnT_n - I_n$$

where $I_n = \sum_k k i_{n,k}$. If $I^{[p-1]}(t,w) = \sum i_{n,k} t^n w^k$ then we have:

$$\begin{cases} I^{[p-1]}(t,w) & = & 1 + tw^{p-1}T(t)^2 I^{[1]}(t,w)\dots I^{[p-1]}(t,w) \\ I^{[p-2]}(t,w) & = & 1 + tw^{p-2}T(t)^3 I^{[1]}(t,w)\dots I^{[p-2]}(t,w) \\ & & \\ & & \\ I^{[2]}(t,w) & = & 1 + tw^{2}T(t)^{p-1} I^{[1]}(t,w) I^{[2]}(t,w) \\ I^{[1]}(t,w) & = & 1 + twT(t)^{p} I^{[1]}(t,w) \end{cases}$$

and by solving the system we get:

$$I^{[p-1]}(t,w) = \frac{1 - w + tw^{p-1}T(t)^2}{1 - w + tw^p T(t)}$$

$$I_n = [t^n] \left. \frac{\partial I^{[p-1]}(t,w)}{\partial w} \right|_{w=1} = -\frac{tT(t)-1}{t} - \frac{1}{tT(t)} =$$

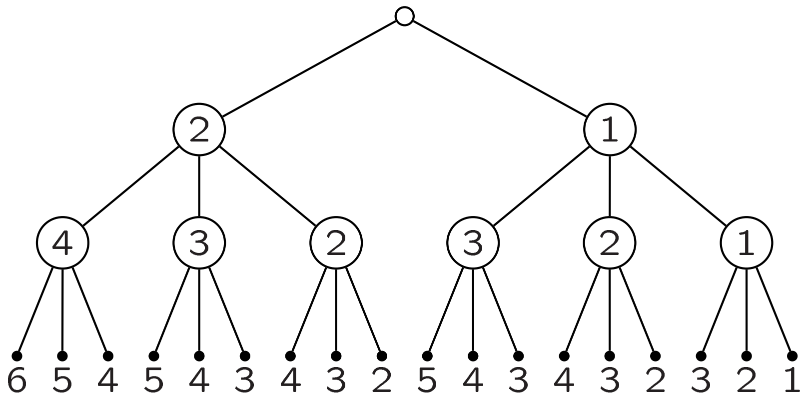
$$= \frac{p(p(n+1)+1)T_{n+1}}{(p+1)((p+1)(n+1)-1)} - T_n$$

For $p = 1$ $I_n = 0$.

Total # of users among $p^{[r]}$-histograms, $r = 1, \ldots, p$: MIN CASE

In the general case, for $p^{[r]}$-histograms, $r = 1, \ldots, p$ we introduce the concept of e-labelled p $(p+1)$-ary trees:

A complete e-labelled 2 ternary tree ($p = 2$)



Theorem: The number $T^{[r]}_{n,k}$ of $p^{[r]}$-histograms of length n, $r = 1, \ldots, p$ corresponding to MIN number of users $= k$ is equal to the number of p $(p+1)$-ary trees with n internal nodes (root excluded) and k marked external nodes. $\mathcal{U}^{MIN}_n = \sum_{k\geq 0} T^{[r]}_{n,k}$. If $\overline{T}(t,w) = \sum_{k\geq 0} T^{[r]}_{n,k} t^n w^k$ then we have:

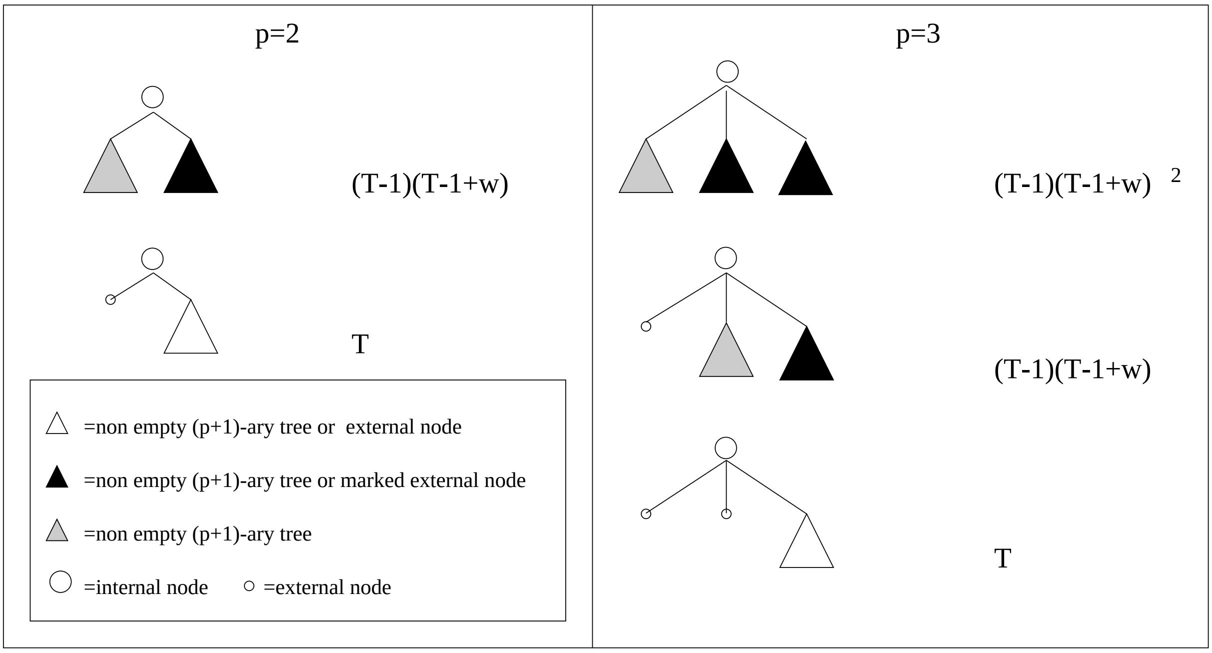
$$\overline{T}(t,w) = T(t,w) + (T(t,w) - 1)\sum_{k=1}^{p-1}(T(t,w) - 1 + w)^k$$

$$\mathcal{U}^{MIN}_n = [t^n]\left.\frac{\partial \overline{T}(t,w)}{\partial w}\right|_{w=1} =$$

$$= [t^n]\frac{1}{T(t)-1} - \frac{p}{pT(t)-p-1} - \frac{1}{tT(t)} =$$

$$= \frac{p(pn+1)(pn-1)}{(p+1)(n+1)}T_n + \frac{p(pn+p+1)}{(p+1)(pn+p+n)}T_{n+1}$$

Counting marked external nodes for p $(p+1)$-ary tree: $T = T(t,w)$.



Total # of users among $p^{[r]}$-histograms, $r = 1, \ldots, p$: MAX CASE

Theorem: The number $i_{n,k}^{[r]}$ of $p^{[r]}$-histograms of length n, $r = 1, \ldots, p$, having a total idle time of lenght k is equivalent to the number of p $(p+1)$-ary tree with n internal nodes (root excluded) and weight k.

We have $\mathcal{U}_n^{MAX} = pn\mathcal{T}_n - \mathcal{I}_n$, where $\mathcal{I}_n = \sum_{k\geq 0} k i_{n,k}^{[r]}$:

$$I(t,w) = \sum_{k\geq 0} i_{n,k}^{[r]} t^n w^k = T(t) I^{[1]}(t,w) \ldots I^{[p-1]}(t,w)$$

where $I^{[1]}, \ldots, I^{[r]}$ are the functions defined before. Hence,

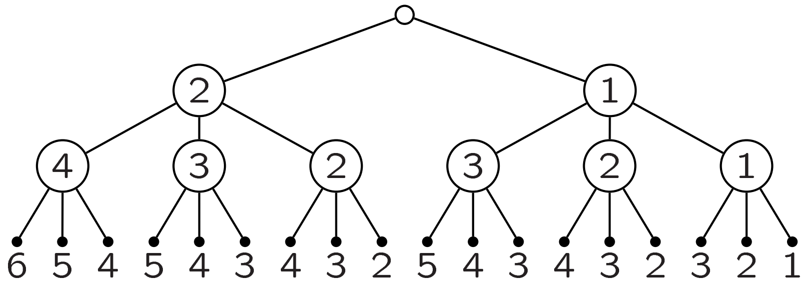
$$\mathcal{I}_n = [t^n] \left. \frac{\partial I(t,w)}{\partial w} \right|_{w=1} =$$

$$= [t^n] - \frac{p}{t} - \frac{1}{t^2 T(t)^2} + \frac{t(p-1)+1}{t^2 T(t)} =$$

$$= \frac{p(p-1)(p(n+2)+1)(n+2)}{(p+1)(p(n+2)+n+1)(p(n+2)+n)} T_{n+2}$$

$$- \frac{p(p-1)(p(n+1)+1)}{(p+1)(p(n+1)+n)} T_{n+1}$$

Total area for $p^{[r]}$-histograms, $r = 1, \ldots, p$



$$A(t,w) = 1 + twA(t,w)A(tw,w)\ldots A(tw^{p-1},w)A(tw^p,w)$$

$$\mathcal{A}(t,w) = A(t,w)A(tw,w)\ldots A(tw^{p-1},w) = \frac{A(t,w)-1}{twA(tw^p,w)}$$

We have:

$$A(t) = \left.\frac{\partial A(t,w)}{\partial w}\right|_{w=1} = \frac{(p-1)T(t)+2}{2p} + \\ + \frac{(p+1)^2}{2p^2(pT(t)-p-1)^2} + \frac{(p+1)(2p+1)}{2p^2(pT(t)-p-1)}$$

$$\mathcal{A}(t) = \left.\frac{\partial \mathcal{A}(t,w)}{\partial w}\right|_{w=1} = \frac{p}{2(p+1)tT(t)} + \frac{1}{2t(pT(t)-p-1)^2} + \\ + \frac{2p+1}{2(p+1)t(pT(t)-p-1)}$$

We have for $p^{[1]}$-histograms:

$$A_n = \frac{1}{2}Y_n - \frac{(2p-1)(n-1)+2p}{2}T_n$$

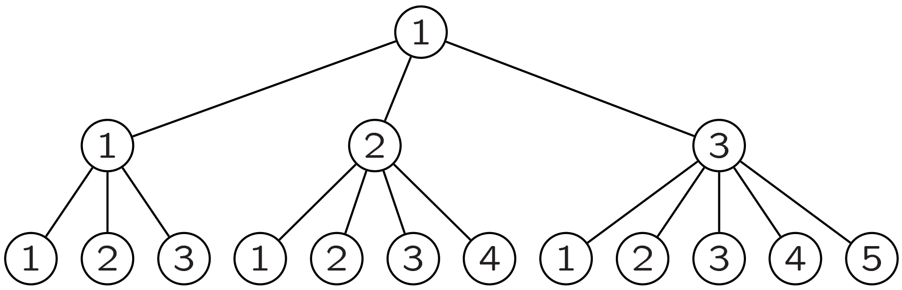
and for $p^{[r]}$-histograms, $r = 1, \ldots, p$:

$$\mathcal{A}_n = \frac{p^2}{2(p+1)^2}Y_{n+1} - \frac{p(pn+p+1)(2p^2(n+1)+pn-n)}{2(p+1)^2(pn+p+n)}T_{n+1}$$

where:

$$Y_n = \sum_{k=0}^{n} \binom{(p+1)k}{k}\binom{(p+1)(n-k)}{n-k}$$

Total area for $p^{[1]}$-histograms



Let $d_{n,k}$ be the number of nodes at level n with label $k+1$ in the generating tree. For example, for $p=2$:

n/k	0	1	2	3	4	5	6
0	1						
1	1	1	1				
2	3	3	3	2	1		
3	12	12	12	9	6	3	1

We have:

$$d_r(t) = p_r(t)T(t) - p_{r-1}(t)$$

$$p_r(t) = [w^r]\frac{1}{1-w+tw^{p+1}}$$

The sum of the labels at level i in all the paths from 0 to n in the generating tree are:

$$\alpha_{i,n} = \sum_{r\geq 0} d_{i,r}(r+1)[t^{n-i}]T(t)^{r+p+1}$$

$$A_n = \sum_{i=0}^{n} \alpha_{i,n}$$

Total area for $p^{[r]}$-histograms, $r = 1, \dots, p$

To compute $\mathcal{A}_n$ we first compute A_n, and then subtract the contribution of the subtree which corresponds to the children of the root labelled $p+1$. This subtree corrisponds to the following specification:

$$\begin{cases} root: & (p+1) \\ rule: & (k) \quad \rightarrow (1)\cdots(k+p-1)(k+p) \end{cases}$$

Finally, we have to subtract the root as many times it is considered.

$$\mathcal{A}_n = A_n - E_{n-1} - T_{n+1}$$

Let $e_{n,k}$ be the number of nodes at level n with label k in the previous generating tree, then

$$E_n = \sum_{i=0}^{n} \epsilon_{i,n}$$

$$\epsilon_{i,n} = \sum_{r \geq 0} e_{i,r}(r+1)[t^{n-i}]T(t)^{r+p+1}$$

The matrix E for $p = 2$.

n/k	0	1	2	3	4	5	6	7	8
0	0	0	1						
1	1	1	1	1	1				
2	5	5	5	4	3	2	1		
3	25	25	25	20	15	10	6	3	1

$$e_r(t) = te_r(t)T(t)^{2p+1} - tp_{r-(2p+1)}(t), \qquad r \geq 2p+1$$

$$e_r(t) = te_r(t)T(t)^{2p+1}, \qquad 0 \leq r \geq 2p, r \neq p$$

$$e_p(t) = te_p(t)T(t)^{2p+1} + 1$$

We have:

$$A(t)=\frac{(p-1)T(t)+2}{2pt}+\frac{(p+1)^2}{2p^2t(pT(t)-p-1)^2}+$$

$$+\frac{(p+1)(2p+1)}{2p^2t(pT(t)-p-1)}$$

$$A_n=\frac{1}{2}Y_{n+1}-\frac{(2p-1)n+2p}{2}T_{n+1}$$

$$\mathcal{A}(t)=\frac{p}{2(p+1)tT(t)}+\frac{1}{2t(pT(t)-p-1)^2}+$$

$$+\frac{2p+1}{2(p+1)t(pT(t)-p-1)}$$

$$\mathcal{A}_n=\frac{p^2}{2(p+1)^2}Y_{n+1}-\frac{p(pn+p+1)(2p^2(n+1)+pn-n)}{2(p+1)^2(pn+p+n)}T_{n+1}$$

where:

$$Y_n=\sum_{k=0}^{n}\binom{(p+1)k}{k}\binom{(p+1)(n-k)}{n-k}$$

Finally we have:

$$AveU_n^{MIN} = \frac{\mathcal{U}_n^{MIN}}{\mathcal{T}_n}, \qquad AveU_n^{MAX} = \frac{\mathcal{U}_n^{MAX}}{\mathcal{T}_n}$$

$$AveW_n^{MIN} = \frac{\mathcal{A}_n}{p\mathcal{U}_n^{MIN}}, \qquad AveW_n^{MAX} = \frac{\mathcal{A}_n}{p\mathcal{U}_n^{MAX}}$$

$$AveQ_n^{MIN=MAX} = \frac{\mathcal{A}_n}{pn\mathcal{T}_n}$$

$p = 1$	$AveU_n^{MIN}$	$AveU_n$	$AveU_n^{MAX}$
$n = 2$	1.500	1.666	2
$n = 4$	2.500	3.066	4
$n = 6$	3.500	4.477	6

$p = 1$	$AveW_n^{MIN}$	$AveW_n$	$AveW_n^{MAX}$
$n = 2$	1.666	1.400	1.250
$n = 4$	2.657	1.942	1.660
$n = 6$	3.432	2.370	2.002

$p = 1$	$AveQ_n$	$AveQ_n^{MIN=MAX}$
$n = 2$	1.166	1.250
$n = 4$	1.488	1.660
$n = 6$	1.768	2.002

Open Questions:

- How much realistic the model is?
- Proving inequalities concerning $AveU_n$, $AveW_n$ and $AveQ_n$
- Finding formulas for $AveU_n$, $AveW_n$ and $AveQ_n$