

Binary words excluding a pattern and proper Riordan Arrays

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Abstract

We study the relation between binary words excluding a pattern and proper Riordan arrays. In particular, we prove necessary and sufficient conditions under which the number of words counted with respect to the number of zeroes and ones bits are related to proper Riordan arrays. We also give formulas for computing the generating functions $(d(x), h(x))$ defining the Riordan array.

1 Introduction

The concept of a *Riordan array* has been introduced in 1991 by Shapiro, Getu, Woan and Woodson [11], with the aim of generalizing the concept of a *renewal array* defined by Rogers [9] in 1978. Their basic idea was to define a class of infinite lower triangular arrays with properties analogous to those of the Pascal triangle. This concept has also been studied by Sprugnoli [14], who pointed out the relevance of these matrices from a theoretical and practical point of view. Later, several new characterizations of Riordan arrays have been given in [4]: the main result in that paper shows that a lower triangular array $d_{n,k}$ is Riordan whenever its generic element $d_{n+1,k+1}$ linearly depends on the elements $d_{r,s}$ lying in a well-defined, but large zone of the array (see Theorem 4.5 and Figure 1 in the present paper). This fact provides a remarkable characterization of many lower triangular arrays for which a recurrence can be given involving elements belonging to the relevant zone.

In this article we study the enumeration of binary words excluding a fixed pattern \mathbf{p} . In particular, we consider the numbers $F_{n,k}^{[\mathbf{p}]}$ of words avoiding \mathbf{p} and containing n zeroes and k ones. These numbers constitute an infinite matrix $\mathcal{F}^{[\mathbf{p}]}$. For example, when $\mathbf{p} = 101$, $\mathcal{F}^{[\mathbf{p}]}$ begins as follows:

$n \setminus k$	0	1	2	3	4	5	6	...
0	1	1	1	1	1	1	1	...
1	1	2	3	4	5	6	7	...
2	1	2	4	7	11	16	22	...
3	<u>1</u>	<u>2</u>	5	<u>10</u>	18	30	47	...
4	1	2	6	13	26	48	83	...
5	<u>1</u>	<u>2</u>	<u>7</u>	16	<u>35</u>	70	131	...
6	1	2	8	19	45	96	192	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

In some of these matrices we can observe a certain regularity. For example, in the above matrix we point out that $F_{4,3}^{[\mathbf{p}]} = F_{3,3}^{[\mathbf{p}]} + F_{3,1}^{[\mathbf{p}]} + F_{3,0}^{[\mathbf{p}]}$ and $F_{6,4}^{[\mathbf{p}]} = F_{5,4}^{[\mathbf{p}]} + F_{5,2}^{[\mathbf{p}]} + F_{5,1}^{[\mathbf{p}]} + F_{5,0}^{[\mathbf{p}]}$. We notice, in fact, that every element in the lower triangular part of the matrix $\mathcal{F}^{[\mathbf{p}]}$ depends in a similar way from the elements in the previous row and the previous columns. This fact connects $\mathcal{F}^{[\mathbf{p}]}$ to the concept of a Riordan array. This connection was originally observed by Emanuele Munarini [8] who proposed the problem to the third author of this paper. We then decided to study, in terms of Riordan arrays, the matrix $\mathcal{R}^{[\mathbf{p}]} = (R_{n,k}^{[\mathbf{p}]})$ such that $R_{n,k}^{[\mathbf{p}]} = F_{n,n-k}^{[\mathbf{p}]}$. For the pattern $\mathbf{p} = 101$ we obtain the following matrix in which every element can be found as a linear combination of the elements in the previous row, starting from the previous column.

$n \backslash k$	0	1	2	3	4	5	6	...
0	1	0	0	0	0	0	0	...
1	2	1	0	0	0	0	0	...
2	4	2	1	0	0	0	0	...
3	<u>10</u>	5	<u>2</u>	<u>1</u>	0	0	0	...
4	26	13	6	2	1	0	0	...
5	70	<u>35</u>	16	<u>7</u>	<u>2</u>	<u>1</u>	0	...
6	192	96	45	19	8	2	1	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

As we will see in Section 4, this corresponds exactly to the concept of a Riordan array.

In the recent literature, Riordan arrays have attracted the attention of various authors and many examples and applications can be found [5, 6, 7, 12, 18, 19, 20]. However, most of them deal with the original concept of Riordan arrays, that is, in the corresponding matrices each element is described by a linear combination of the elements in the previous row, starting from the previous column (see Theorem 4.1). Often, a Riordan array has a combinatorial interpretation and the relation with the elements in the previous row translates into a way to construct a class of combinatorial objects of a certain size $n + 1$, starting from the objects of size n . As we will see in this paper, the connection between the language of words avoiding a given pattern and Riordan arrays corresponds to matrices which are naturally defined by recurrence relations involving elements belonging to the grey zone in Figure 1 and, therefore, gives rise to an entire class of new examples of Riordan arrays which follow the new characterization [4]. We believe this is interesting in the theory of Riordan arrays. Moreover, Theorem 4.1 also implies the existence of an algebraic relation involving only the elements in the previous row which, however, doesn't seem to translate naturally into a combinatorial relation between objects of size $n + 1$ and n . As we will discuss in the Conclusions, this opens new questions which are worth to be further examined.

The paper is organized in the following way: in Section 2, we describe the problem of finding the generating function of the words excluding a pattern on a generic alphabet. In Section 3, we develop the problem for binary strings by using bivariate generating functions. In Section 4, we give the definitions and the characterizations of Riordan arrays. In Section 5, we give a classification of binary patterns under which $\mathcal{R}^{[p]}$ is a Riordan matrix. Finally, in Section 6, we illustrate some methods to find a characterization of the matrix $\mathcal{R}^{[p]}$ in terms of generating functions.

2 Words excluding a pattern

Let \mathcal{A} be any alphabet. The enumeration of words on \mathcal{A}^* which do not contain a fixed pattern $\mathbf{p} = p_0 \cdots p_{h-1}$ has been studied in terms of generating functions from several authors (see, e.g., [10]). If \mathcal{S} denotes the language of words with no occurrence of \mathbf{p} , the problem is to determine the generating function $S(z)$ counting the number of words with respect to their length. A nice explicit construction is due to Guibas and Odlyzko [2] (see also [3] for more details).

The fundamental notion is that of an *autocorrelation vector*. For a given \mathbf{p} , this vector of bits $c = (c_0, \dots, c_{h-1})$ is most conveniently defined in terms of Iverson's bracket notation (for a predicate P , the expression $\llbracket P \rrbracket$ has value 1 if P is true and 0 otherwise) as

$$c_i = \llbracket p_0 p_1 \cdots p_{h-1-i} = p_i p_{i+1} \cdots p_{h-1} \rrbracket.$$

In other words, the bit c_i is determined by shifting \mathbf{p} right by i positions and setting a $c_i = 1$ iff the remaining letters match the original. For instance, with $\mathbf{p} = \text{aabbaa}$, one has

a	a	b	b	a	a	Tails					
a	a	b	b	a	a					1	
	a	a	b	b	a	a				0	
		a	a	b	b	a	a			0	
			a	a	b	b	a	a		0	
				a	a	b	b	a	a	1	
					a	a	b	b	a	a	1

The autocorrelation is then $c = (1, 0, 0, 0, 1, 1)$. The *autocorrelation polynomial* is defined as

$$c(z) = \sum_{j=0}^{h-1} c_j z^j,$$

that is, we mark with z^j the tails of the pattern of length j and $c_j = 1$. For the sample pattern, this gives $c(z) = 1 + z^4 + z^5$.

Let \mathcal{T} be the language of words that end with \mathbf{p} , but have no other occurrence of \mathbf{p} . First, by appending a letter to a word of \mathcal{S} , one finds a non-empty word either in \mathcal{S} or \mathcal{T} , so that,

$$\mathcal{S} \cup \mathcal{T} = \{\epsilon\} \cup \mathcal{S} \times \mathcal{A}. \quad (2.1)$$

Next, appending a copy of the word \mathbf{p} to a word in \mathcal{S} may only give words that contain \mathbf{p} at or near the end. Precisely, the decomposition based on the leftmost occurrence of \mathbf{p} in \mathcal{S} is

$$\mathcal{S} \times \{\mathbf{p}\} = \mathcal{T} \times \sum_{c_i \neq 0} \{p_i p_{i+1} \dots p_{h-1}\}. \quad (2.2)$$

By using the symbolic method (see, e.g. [10]) we can translate the system (2.1), (2.2) into generating functions as follows:

Theorem 2.1 *The generating function counting the number S_n of words of length n not containing the pattern \mathbf{p} is*

$$S(z) = \frac{c(z)}{z^h + (1 - mz)c(z)} \quad (2.3)$$

where m is the alphabet cardinality, $h = |\mathbf{p}|$ the pattern length, and $c(z)$ the autocorrelation polynomial, $c(z) = \sum_i c_i z^i$. Moreover, the generating function counting the number T_n of words of length n containing \mathbf{p} only once at the end is

$$T(z) = \frac{z^h}{z^h + (1 - mz)c(z)}. \quad (2.4)$$

If $[z^n]$ denotes the *coefficient of operator* we therefore have $S_n = [z^n]S(z)$ and $T_n = [z^n]T(z)$.

3 Binary words excluding a pattern

In this paper we are interested in studying binary words excluding a pattern $\mathbf{p} = p_0 \dots p_{h-1} \in \{0, 1\}^h$ with respect to the number of zeroes and ones. By using the indeterminates x and y to denote these numbers, the system (2.1), (2.2) can be easily transformed into the following bivariate generating function:

$$F^{[\mathbf{p}]}(x, y) = \frac{C^{[\mathbf{p}]}(x, y)}{(1 - x - y)C^{[\mathbf{p}]}(x, y) + x^{n_1^{\mathbf{p}}} y^{n_0^{\mathbf{p}}}}, \quad (3.1)$$

and $F_{n,k}^{[\mathbf{p}]} = [x^n y^k]F^{[\mathbf{p}]}(x, y)$ denotes the number of words excluding the pattern with n bits 1 and k bits 0. Obviously $F^{[\mathbf{p}]}(z, z)$ equals the function $S(z)$ of the previous section with $m = 2$. For all $0 \leq i \leq j \leq h$ we define the functions counting the number of zeroes and ones in $p_i \dots p_j$ as follows:

$$N_0^{\mathbf{p}}(i, j) = \sum_{\lambda=i}^j \delta_{p_\lambda, 0},$$

$$N_1^{\mathbf{p}}(i, j) = \sum_{\lambda=i}^j \delta_{p_\lambda, 1},$$

where the terms $\delta_{p_\lambda, 0}$ and $\delta_{p_\lambda, 1}$ are Kronecker deltas. For the sake of simplicity, we use the following abbreviations:

$$n_0^{\mathbf{p}} = N_0^{\mathbf{p}}(0, h-1),$$

$$n_1^{\mathbf{p}} = N_1^{\mathbf{p}}(0, h-1),$$

$$n_0^{\mathbf{p}}(i) = N_0^{\mathbf{p}}(h-1-i, h-1),$$

$$n_1^{\mathbf{p}}(i) = N_1^{\mathbf{p}}(h-1-i, h-1),$$

where $n_0^{\mathbf{p}}(i)$ and $n_1^{\mathbf{p}}(i)$ count respectively the number of zeroes and ones in the tail of length i . Therefore, by using these notations we can write the autocorrelation polynomial as follows:

$$C^{[\mathbf{p}]}(x, y) = 1 + \sum_{i=1}^{h-1} c_i x^{n_1^{\mathbf{p}}(i)} y^{(i-n_1^{\mathbf{p}}(i))}. \quad (3.2)$$

In the sequel, if we fix the pattern \mathbf{p} , we can omit the superscript \mathbf{p} . From formulas (3.1) and (3.2) we have:

$$\left[(1-x-y) \left(1 + \sum_{i=1}^{h-1} c_i x^{n_1(i)} y^{(i-n_1(i))} \right) + x^{n_1} y^{n_0} \right] F(x, y) = C(x, y).$$

We wish to study the relation between the elements of the array \mathcal{F} associated to $F(x, y)$, so we use the “coefficient of” operator and extract the $[x^{n+1}y^{k+1}]$ coefficient from the previous relation:

$$\begin{aligned} [x^{n+1}y^{k+1}] C(x, y) &= [x^{n+1}y^{k+1}] \left(\left(1 + \sum_{i=1}^{h-1} c_i x^{n_1(i)} y^{(i-n_1(i))} \right) F(x, y) \right) \\ &\quad - [x^{n+1}y^{k+1}] \left(x \left(1 + \sum_{i=1}^{h-1} c_i x^{n_1(i)} y^{(i-n_1(i))} \right) F(x, y) \right) \\ &\quad - [x^{n+1}y^{k+1}] \left(y \left(1 + \sum_{i=1}^{h-1} c_i x^{n_1(i)} y^{(i-n_1(i))} \right) F(x, y) \right) \\ &\quad + [x^{n+1-n_1}y^{k+1-n_0}] F(x, y). \end{aligned}$$

We have

$$\begin{aligned} \sum_{i=0}^{h-1} c_i \delta_{n+1, n_1(i)} \delta_{k+1, i-n_1(i)} &= [x^{n+1}y^{k+1}] F(x, y) + \sum_{i=1}^{h-1} c_i [x^{n+1-n_1(i)}y^{k+1-i+n_1(i)}] F(x, y) \\ &\quad - [x^n y^{k+1}] F(x, y) - \sum_{i=1}^{h-1} c_i [x^{n-n_1(i)}y^{k+1-i+n_1(i)}] F(x, y) \\ &\quad - [x^{n+1}y^k] F(x, y) - \sum_{i=1}^{h-1} c_i [x^{n+1-n_1(i)}y^{k-i+n_1(i)}] F(x, y) \\ &\quad + [x^{n+1-n_1}y^{k+1-n_0}] F(x, y), \end{aligned}$$

and consequently

$$\begin{aligned} F_{n+1, k+1} &= F_{n, k+1} + F_{n+1, k} - F_{n+1-n_1, k+1-n_0} + \sum_{i=0}^{h-1} c_i \delta_{n+1, n_1(i)} \delta_{k+1, i-n_1(i)} \\ &\quad - \sum_{i=1}^{h-1} c_i (F_{n+1-n_1(i), k+1-i+n_1(i)} - F_{n-n_1(i), k+1-i+n_1(i)} - F_{n+1-n_1(i), k-i+n_1(i)}), \quad (3.3) \end{aligned}$$

We now define a new array $\mathcal{R} = (R_{n,k})$ from the lower triangular part of the array \mathcal{F} as follows:

$$R_{n,k} = F_{n,n-k} \quad \text{with } k \leq n.$$

The recurrence relation (3.3) becomes:

$$\begin{aligned} R_{n+1,n-k} &= R_{n,n-k-1} + R_{n+1,n-k+1} - R_{n+1-n_1,n-k+n_0-n_1} + \sum_{i=0}^{h-1} c_i \delta_{n+1,n_1(i)} \delta_{n-k,n_1(i)-n_0(i)} \\ &- \sum_{i=1}^{h-1} c_i \left(R_{n+1-n_1(i),n-n_1(i)-k+i-n_1(i)} - R_{n-n_1(i),n-n_1(i)-k-1+i-n_1(i)} - R_{n+1-n_1(i),n+1-n_1(i)-k+i-n_1(i)} \right). \end{aligned}$$

If we change the variable $n - k$ into $j + 1$, we have the proof of the following theorem:

Theorem 3.1 *Given a pattern $\mathbf{p} = p_0 \dots p_{h-1} \in \{0, 1\}^h$, let $\mathcal{R} = (R_{n,k})$ be the matrix defined as $R_{n,k} = F_{n,n-k} = [x^n y^{n-k}]F(x, y)$ with $k \leq n$, that is, $R_{n,k}$ counts the number of words of length $2n - k$ and with n bits equal to 1 avoiding \mathbf{p} . Then the following recurrence relation holds true:*

$$\begin{aligned} R_{n+1,j+1} &= R_{n,j} + R_{n+1,j+2} - R_{n+1-n_1,j+1+n_0-n_1} + \sum_{i=0}^{h-1} c_i \delta_{n+1,n_1(i)} \delta_{j+1,n_1(i)-n_0(i)} \\ &- \sum_{i=1}^{h-1} c_i \left(R_{n+1-n_1(i),j+1+i-2n_1(i)} - R_{n-n_1(i),j+i-2n_1(i)} - R_{n+1-n_1(i),j+2+i-2n_1(i)} \right). \end{aligned} \quad (3.4)$$

Example 3.1 *Let us study the case $\mathbf{p} = 01000$, for which we have $C(x, y) = 1 + xy^3$ and hence*

$$F(x, y) = \frac{1 + xy^3}{(1 - x - y)(1 + xy^3) + xy^4}.$$

In this case the relation (3.4) reduces to

$$R_{n+1,j+1} = R_{n,j} + R_{n+1,j+2} + R_{n-1,j+2} - R_{n,j+3},$$

and the array \mathcal{R} associated to \mathbf{p} is as follows:

$n \setminus j$	0	1	2	3	4	5	6	7	...
0	1	0	0	0	0	0	0	0	...
1	2	1	0	0	0	0	0	0	...
2	6	3	1	0	0	0	0	0	...
3	20	10	4	1	0	0	0	0	...
4	66	35	15	5	1	0	0	0	...
5	222	121	56	21	6	1	0	0	...
6	756	420	204	84	28	7	1	0	...
7	2598	1464	736	323	120	36	8	1	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

In this paper we wish to find for which patterns the matrix \mathcal{R} is a Riordan array. To this purpose, in the next section we recall the main properties of Riordan arrays. For details and proofs the reader can refer to the paper [4].

4 Riordan Arrays

A *Riordan array* is a pair $(d(t), h(t))$ in which $d(t)$ and $h(t)$ are formal power series such that $d(0) \neq 0$; if $h(0) \neq 0$, the Riordan array is called *proper*. The pair defines an infinite, lower triangular array $\{d_{n,k}\}_{n,k \in \mathbb{N}}$ where:

$$d_{n,k} = [t^n]d(t)(th(t))^k.$$

From this definition, it easily follows that $d(t)(th(t))^k$ is the generating function of column k in the array. Moreover,

$$\frac{d(t)}{1 - twh(t)}$$

is the bivariate generating function of the triangle. The Riordan array theory allows us to find properties concerning these matrices; for example, we have:

$$\sum_{k=0}^n d_{n,k} f_k = [t^n]d(t)f(th(t)),$$

for every sequence f_k having $f(t)$ as its generating function. A description of the Riordan array theory together with many examples, can be found in Shapiro et al. [11] or in Sprugnoli [14]. Rogers [9] proved the following, fundamental characterization of proper Riordan arrays:

Theorem 4.1 *An array $\{d_{n,k}\}_{n,k \in \mathbb{N}}$ is a proper Riordan array if and only if there exists a sequence $A = \{a_i\}_{i \in \mathbb{N}}$ with $a_0 \neq 0$ such that every element $d_{n+1,k+1}$ (not lying in column 0 or row 0) can be expressed as a linear combination with coefficients in A of the elements in the preceding row, starting from the preceding column, i.e.:*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \dots. \quad (4.1)$$

The sum in (4.1) is actually finite because $d_{n,k} = 0, \forall k > n$. Sequence A , called the A -sequence of the Riordan array, is characteristic in the sense that it determines (and is determined by) the function $h(t)$. If $A(t)$ is the generating function of the A -sequence, it can be proven (see Sprugnoli [14]) that $h(t)$ is the solution of the functional equation:

$$h(t) = A(th(t)). \quad (4.2)$$

The A -sequence does not completely characterize a proper Riordan array $(d(t), h(t))$ because the function $d(t)$ is independent of $A(t)$. In [4] the following new characterizations have been proved:

Theorem 4.2 *Let $\{d_{n,k}\}_{n,k \in \mathbb{N}}$ be any infinite lower triangular array with $d_{n,n} \neq 0, \forall n \in \mathbb{N}$ (in particular, let it be a proper Riordan array); then a unique sequence $Z = \{z_0, z_1, z_2, \dots\}$ exists such that every element in column 0 can be expressed as a linear combination of all the elements in the preceding row, i.e.:*

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \dots.$$

The Z -sequence characterizes column 0, while the A -sequence characterizes all the other columns. The triple $(d_0, Z(t), A(t))$ characterizes a proper Riordan array:

Theorem 4.3 *Let $(d(t), h(t))$ be a proper Riordan array and let $Z(t)$ be the generating function of its Z -sequence; then:*

$$d(t) = \frac{d_0}{1 - tZ(th(t))}. \quad (4.3)$$

The relation can be inverted and this gives us a formula for the Z -sequence:

$$Z(y) = \left[\frac{d(t) - d_0}{td(t)} \Big|_{t = yh(t)^{-1}} \right]. \quad (4.4)$$

The following theorems, proved in [4], show that we can characterize a Riordan array by means of an A -matrix, rather than by a simple A -sequence.

Theorem 4.4 *A lower triangular array $\{d_{n,k}\}_{n,k}$ is Riordan if and only if there exists another array $A = \{\alpha_{i,j}\}_{i,j \in \mathbb{N}}$, with $\alpha_{0,0} \neq 0$, such that every $d_{n+1,k+1}$ ($n, k \geq 0$) can be expressed as:*

$$d_{n+1,k+1} = \sum_{i \geq 0} \sum_{j \geq 0} \alpha_{i,j} d_{n-i,k+j}. \quad (4.5)$$

However, while the A -sequence is unique for a given Riordan array, the A -matrix is not.

The linear dependence of the generic element $d_{n+1,k+1}$ can be extended to elements on its own row, starting from $d_{n+1,k+2}$. In fact, we can prove the following characterization:

Theorem 4.5 *A lower triangular array $\{d_{n,k}\}_{n,k \in \mathbb{N}}$ is Riordan if and only if there exist another array $\{\alpha_{i,j}\}_{i,j \in \mathbb{N}}$, with $\alpha_{0,0} \neq 0$, and s sequences $\{\rho_j^{[i]}\}_{j \in \mathbb{N}}$ ($i = 1, 2, \dots, s$) such that:*

$$d_{n+1,k+1} = \sum_{i \geq 0} \sum_{j \geq 0} \alpha_{i,j} d_{n-i,k+j} + \sum_{i=1}^s \sum_{j \geq 0} \rho_j^{[i]} d_{n+i,k+i+j+1}. \quad (4.6)$$

In Figure 1, we try to give a graphic representation of the zones from which the generic element $d_{n+1,k+1}$ (denoted by a small disk or bullet) is allowed to depend, so that the array is Riordan. The two zones correspond to Theorems 4.4 and 4.5, and the only restrictions are that $\alpha_{0,0} \neq 0$ and that the number of rows below row n is finite. As previously noted, the A -sequence and the function $h(t)$ of a Riordan array are strictly

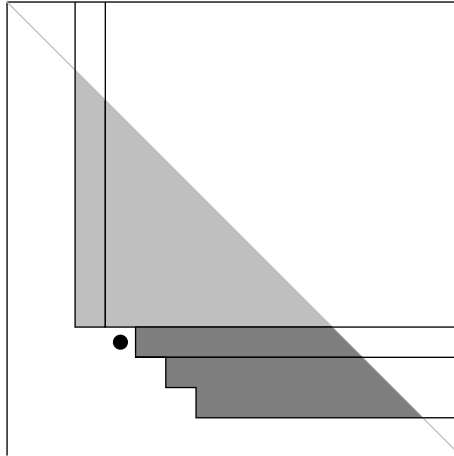


Figure 1: The zones which $d_{n+1,k+1}$ can depend on.

related to each other. This fact allows us to think that $h(t)$ can be deduced from the A -matrix $\{\alpha_{i,j}\}_{i,j \in \mathbb{N}}$ and the set of sequences $\{\rho_j^{[i]}\}_{j \in \mathbb{N}}$ for $i = 0, 1, \dots, s$. So, after having found the function $h(t)$, we can also find the A -sequence by determining its generating function $A(t)$. Almost always, $d_{n+1,k+1}$ only depends on the elements of a finite number of rows above it; therefore, instead of treating a global generating function for the A -matrix, let us examine a sequence of generating functions $P^{[0]}(t), P^{[1]}(t), P^{[2]}(t), \dots$ corresponding to the rows $0, 1, 2, \dots$ of the A -matrix, i.e.:

$$\begin{aligned} P^{[0]}(t) &= \alpha_{0,0} + \alpha_{0,1}t + \alpha_{0,2}t^2 + \alpha_{0,3}t^3 + \dots \\ P^{[1]}(t) &= \alpha_{1,0} + \alpha_{1,1}t + \alpha_{1,2}t^2 + \alpha_{1,3}t^3 + \dots \end{aligned}$$

and so on. Moreover, let $Q^{[i]}(t)$ be the generating function for the sequence $\{\rho_j^{[i]}\}_{j \in \mathbb{N}}$. Thus we have:

Theorem 4.6 *If $\{d_{n,k}\}_{n,k \in \mathbb{N}}$ is a Riordan array whose generic element $d_{n+1,k+1}$ is defined by formula (4.6) through the A -matrix $\{\alpha_{i,j}\}_{i,j \in \mathbb{N}}$ and the set of sequences $\{\rho_j^{[i]}\}_{j \in \mathbb{N}}$, $i = 1, 2, \dots, s$, then the functions $h(t)$ and $A(t)$ for $\{d_{n,k}\}$ are given by the following implicit expressions:*

$$h(t) = \sum_{i \geq 0} t^i P^{[i]}(th(t)) + \sum_{i=1}^s th(t)^{i+1} Q^{[i]}(th(t)), \quad (4.7)$$

$$A(t) = \sum_{i \geq 0} t^i A(t)^{-i} P^{[i]}(t) + t \sum_{i=1}^s A(t)^i Q^{[i]}(t). \quad (4.8)$$

Definition 5.1 Given a pattern $\mathbf{p} = p_0 \dots p_{h-1} \in \{0, 1\}^h$, let c be its associated autocorrelation vector. We call \mathbf{p} a $+$ monotonic pattern if $\forall i \geq 0$ with $c_i = 1$ we have:

$$n_0(i) \geq n_1(i). \quad (5.2)$$

In the same way, we call \mathbf{p} a $-$ monotonic pattern if $\forall i \geq 0$ with $c_i = 1$ we have:

$$n_0(i) \leq n_1(i). \quad (5.3)$$

We call \mathbf{p} a $=$ monotonic pattern if it is both $+$ monotonic and $-$ monotonic.

Example 5.3 Let $\mathbf{p} = 1110000111$ be a pattern; \mathbf{p} is not a $=$ monotonic pattern because:

$$\begin{array}{r|l} 1110000111 & n_0 < n_1 \\ 1110000111 & \\ 111 & 0000111 \quad n_0(7) > n_1(7) \\ 11 & 10000111 \quad n_0(8) = n_1(8) \\ 1 & 110000111 \quad n_0(9) < n_1(9) \end{array}$$

Theorem 5.1 Given a pattern $\mathbf{p} = p_0 \dots p_{h-1} \in \{0, 1\}^h$, let $\bar{\mathbf{p}} = \bar{p}_0 \dots \bar{p}_{h-1}$ be the pattern with $\bar{p}_i = 1 - p_i, \forall i \in \{0, \dots, h-1\}$. Then we have:

a) $\mathcal{R}^{[\mathbf{p}]}$ is a Riordan array and $\mathcal{R}^{[\bar{\mathbf{p}}]}$ is not $\Leftrightarrow \mathbf{p}$ is a $+$ monotonic pattern and

- $\exists i > 0$ such that $c_i = 1$ and $n_0^{\mathbf{p}}(i) - n_1^{\mathbf{p}}(i) > 0$ and $n_0^{\mathbf{p}} - n_1^{\mathbf{p}} \geq -1$;

or, alternatively

- $\forall i > 0$ such that $c_i = 1$, we have $n_0^{\mathbf{p}}(i) - n_1^{\mathbf{p}}(i) = 0$ and $n_0^{\mathbf{p}} - n_1^{\mathbf{p}} > 1$;

b) $\mathcal{R}^{[\bar{\mathbf{p}}]}$ is a Riordan array and $\mathcal{R}^{[\mathbf{p}]}$ is not $\Leftrightarrow \mathbf{p}$ is a $-$ monotonic pattern and

- $\exists i > 0$ such that $c_i = 1$ and $n_1^{\mathbf{p}}(i) - n_0^{\mathbf{p}}(i) > 0$ and $n_1^{\mathbf{p}} - n_0^{\mathbf{p}} \geq -1$;

or, alternatively

- $\forall i > 0$ such that $c_i = 1$, we have $n_1^{\mathbf{p}}(i) - n_0^{\mathbf{p}}(i) = 0$ and $n_1^{\mathbf{p}} - n_0^{\mathbf{p}} > 1$;

c) $\mathcal{R}^{[\mathbf{p}]}$ and $\mathcal{R}^{[\bar{\mathbf{p}}]}$ are both Riordan arrays $\Leftrightarrow \mathbf{p}$ is a $=$ monotonic pattern and $|n_1^{\mathbf{p}} - n_0^{\mathbf{p}}| \in \{0, 1\}$.

Proof: We notice that \mathbf{p} and $\bar{\mathbf{p}}$ have the same autocorrelation vector (c_0, \dots, c_{h-1}) , whereas the autocorrelation polynomials have the exponents of x exchanged with the exponents of y . We have seen that the array $\mathcal{R}^{[\mathbf{p}]}$ satisfies the following recurrence relation:

$$\begin{aligned} R_{n+1, j+1} &= R_{n, j} + R_{n+1, j+2} - R_{n+1-n_1^{\mathbf{p}}, j+1+n_0^{\mathbf{p}}-n_1^{\mathbf{p}}} + \sum_{i=0}^{h-1} c_i \delta_{n+1, n_1^{\mathbf{p}}(i)} \delta_{j+1, n_1^{\mathbf{p}}(i)-n_0^{\mathbf{p}}(i)} \\ &\quad - \sum_{i=1}^{h-1} c_i \left(R_{n+1-n_1^{\mathbf{p}}(i), j+1+i-2n_1^{\mathbf{p}}(i)} - R_{n-n_1^{\mathbf{p}}(i), j+i-2n_1^{\mathbf{p}}(i)} - R_{n+1-n_1^{\mathbf{p}}(i), j+2+i-2n_1^{\mathbf{p}}(i)} \right), \end{aligned} \quad (5.4)$$

whereas for $\mathcal{R}^{[\bar{\mathbf{p}}]}$ we have

$$\begin{aligned} \bar{R}_{n+1, j+1} &= \bar{R}_{n, j} + \bar{R}_{n+1, j+2} - \bar{R}_{n+1-n_0^{\mathbf{p}}, j+1+n_1^{\mathbf{p}}-n_0^{\mathbf{p}}} + \sum_{i=0}^{h-1} c_i \delta_{n+1, n_0^{\mathbf{p}}(i)} \delta_{j+1, n_0^{\mathbf{p}}(i)-n_1^{\mathbf{p}}(i)} \\ &\quad - \sum_{i=1}^{h-1} c_i \left(\bar{R}_{n+1-n_0^{\mathbf{p}}(i), j+1+i-2n_0^{\mathbf{p}}(i)} - \bar{R}_{n-n_0^{\mathbf{p}}(i), j+i-2n_0^{\mathbf{p}}(i)} - \bar{R}_{n+1-n_0^{\mathbf{p}}(i), j+2+i-2n_0^{\mathbf{p}}(i)} \right). \end{aligned} \quad (5.5)$$

Let us examine case a). If $\mathcal{R}^{[\mathbf{p}]}$ is a Riordan array and $\mathcal{R}^{[\bar{\mathbf{p}}]}$ is not Riordan, then the recurrence relation (5.4) must satisfy the conditions exposed in Theorem 4.5. This means that the sum of the Kronecker deltas in (5.4) is equal to zero $\forall n, j \in \mathbb{N}$ with $j \leq n$, that is, the product $c_i \delta_{n+1, n_1^{\mathbf{p}}(i)} \delta_{j+1, n_1^{\mathbf{p}}(i) - n_0^{\mathbf{p}}(i)}$ is equal to zero $\forall i = 0, \dots, h-1$ and $\forall n, j \in \mathbb{N}$. This result depends on the number of zeroes and ones in the tails of length i ; in particular, since we want the product to be zero $\forall n, j \in \mathbb{N}$, we need $n_1^{\mathbf{p}}(i) - n_0^{\mathbf{p}}(i) \leq 0$ $\forall i$ with $c_i = 1$, that is, we need \mathbf{p} to be +monotonic. Let us now examine the other terms of relation (5.4). The elements within the second sum in the right hand side of (5.4) depend only on the value of $i - 2n_1^{\mathbf{p}}(i)$, since when we fix a pattern the value $n_0^{\mathbf{p}} - n_1^{\mathbf{p}}$ remains constant. However, since \mathbf{p} is +monotonic $i - 2n_1^{\mathbf{p}}(i) = n_1^{\mathbf{p}}(i) + n_0^{\mathbf{p}}(i) - 2n_1^{\mathbf{p}}(i) = n_0^{\mathbf{p}}(i) - n_1^{\mathbf{p}}(i) \geq 0$ and the elements of the sum are all at the right hand side of $R_{n+1, j+1}$, according to the fact that $\mathcal{R}^{[\mathbf{p}]}$ is Riordan. The only term which depends on the pattern and that remains to examine is $R_{n+1-n_1^{\mathbf{p}}, j+1+n_0^{\mathbf{p}}-n_1^{\mathbf{p}}}$: since $\mathcal{R}^{[\mathbf{p}]}$ is Riordan we have $j+1+n_0^{\mathbf{p}}-n_1^{\mathbf{p}} \geq j$, that is, $n_0^{\mathbf{p}} - n_1^{\mathbf{p}} \geq -1$. Then, if $\mathcal{R}^{[\bar{\mathbf{p}}]}$ is a Riordan array we need \mathbf{p} to be +monotonic and $n_0^{\mathbf{p}} - n_1^{\mathbf{p}} \geq -1$. Let us now examine $\mathcal{R}^{[\bar{\mathbf{p}}]}$ by distinguishing two cases:

- $\exists i > 0$ such that $c_i = 1$ and $n_0^{\mathbf{p}}(i) - n_1^{\mathbf{p}}(i) > 0$. Then the sum of the Kronecker deltas in (5.5) is different from zero for $n+1 = n_0^{\mathbf{p}}(i)$ and $j+1 = n_0^{\mathbf{p}}(i) - n_1^{\mathbf{p}}(i)$, according to the fact that $\mathcal{R}^{[\bar{\mathbf{p}}]}$ is not Riordan.
- $\forall i > 0$ such that $c_i = 1$ we have $n_0^{\mathbf{p}}(i) - n_1^{\mathbf{p}}(i) = 0$. This means that the sum with the Kronecker deltas in (5.4) is also equal to zero. Moreover, $i - 2n_0^{\mathbf{p}}(i) = 0$ $\forall i$ with $c_i = 1$, hence all the elements within the first sum of (5.5) are at the right hand side of $R_{n+1, j+1}$. Since $\mathcal{R}^{[\bar{\mathbf{p}}]}$ is not Riordan we need $j+1+n_1^{\mathbf{p}}-n_0^{\mathbf{p}} < j$, that is, $n_0^{\mathbf{p}} - n_1^{\mathbf{p}} > 1$.

The converse of case a), can be proved by following the inverse procedure used for the previous proof. Case b) is analogous and case c) follows immediately from a) and b). \blacksquare

Example 5.4 Let us take into consideration $\mathbf{p} = 0100100$. In this case, the matrix $\mathcal{R}^{[\mathbf{p}]}$ satisfies the following relation:

$$R_{n+1, j+1} = R_{n, j} + R_{n+1, j+2} + R_{n-1, j+1} - R_{n, j+2} + R_{n, j+3} + R_{n-2, j+2} - R_{n-1, j+3}$$

$n \setminus j$	0	1	2	3	4	...
0	1	0	0	0	0	...
1	2	1	0	0	0	...
2	6	3	1	0	0	...
3	20	10	4	1	0	...
4	70	35	15	5	1	...
5	248	126	56	21	6	...
6	894	457	210	84	28	...
7	3269	1674	786	330	120	...
8	12072	6189	2947	1280	495	...
...

As illustrated in the figure, the element $R_{n+1, j+1}$ depends on elements belonging to the grey zone in Figure 1, and $\mathcal{R}^{[\mathbf{p}]}$ is a Riordan array. In fact, \mathbf{p} is +monotonic and $n_0^{\mathbf{p}} - n_1^{\mathbf{p}} = 3$. Moreover, since $n_0^{\mathbf{p}}(3) - n_1^{\mathbf{p}}(3) = 2$, $\mathcal{R}^{[\bar{\mathbf{p}}]}$ is not a Riordan array.

Example 5.5 When $\mathbf{p} = 1011101$ we obtain the following recurrence:

$$R_{n+1, j+1} = R_{n, j} + R_{n+1, j+2} - R_{n-2, j-1} + R_{n-3, j-2} + R_{n-2, j} - R_{n-3, j-1} + R_{n-3, j} + \delta_{n+1, 3} \delta_{j+1, 2} + \delta_{n+1, 4} \delta_{j+1, 2}$$

6 Finding $d(t)$ and $h(t)$

As we have seen in Section 3, the generating function of binary words not containing a pattern \mathbf{p} is:

$$F^{[\mathbf{p}]}(x, y) = \frac{C(x, y)}{(1 - x - y)C(x, y) + x^{n_1^{\mathbf{p}}} y^{n_0^{\mathbf{p}}}}$$

to which we associate the array \mathcal{F} . If we indicate $F^{[\mathbf{p}]}(x, y)$ and $F^{[\bar{\mathbf{p}}]}(x, y)$ with $F(x, y)$ and $\bar{F}(x, y)$ respectively, we note that $\bar{F}(x, y) = F(y, x)$. If $R(x, y) = \sum_{n \geq 0} \sum_{j \geq 0} R_{n,k} x^n y^k$ with $R_{n,k} = F_{n,n-k}$, then we

have:

$$\begin{aligned} F(x, y) &= \sum_{n \geq 0} \left(\sum_{k=0} F_{n,k} x^n y^k + \sum_{n \geq 0} \left(\sum_{k > n} F_{n,k} y^k \right) x^n \right) \\ &= \sum_{n \geq 0} \left(\sum_{j \geq 0} R_{n,j} y^{n-j} x^n + \sum_{n \geq 0} \left(\sum_{k > n} F_{n,k} y^k \right) x^n \right) \\ &= R(xy, \frac{1}{y}) + \sum_{n \geq 0} \left(\sum_{k \geq n} F_{n,k} y^k \right) x^n - \sum_{n \geq 0} F_{n,n} y^n x^n \\ &= R(xy, \frac{1}{y}) + \sum_{n \geq 0} \left(\sum_{k \geq n} \bar{F}_{k,n} y^k \right) x^n - d(xy). \end{aligned}$$

We now substitute $j = k - n \geq 0$ in the second sum, thus obtaining:

$$\begin{aligned} F(x, y) &= R(xy, \frac{1}{y}) + \sum_{j \geq 0} \sum_{n \geq 0} (\bar{F}_{k, k-j} y^k x^{k-j}) - d(xy) \\ &= R(xy, \frac{1}{y}) + \bar{R}(xy, \frac{1}{x}) - d(xy). \end{aligned}$$

Using the same idea for $\bar{F}(x, y)$, we have the following system of equations:

$$\begin{cases} F(x, y) = R(xy, \frac{1}{y}) + \bar{R}(xy, \frac{1}{x}) - d(xy) \\ \bar{F}(x, y) = \bar{R}(xy, \frac{1}{y}) + R(xy, \frac{1}{x}) - d(xy) \end{cases} \quad (6.1)$$

When \mathbf{p} is a $=$ monotonic pattern and $|n_1^{\mathbf{p}} - n_0^{\mathbf{p}}| \in \{0, 1\}$, then both the transformed arrays are Riordan, therefore it is possible to make the following substitution:

$$\begin{aligned} R(x, y) &= \frac{d(x)}{1 - xyh(x)}, \\ \bar{R}(x, y) &= \frac{d(x)}{1 - xy\bar{h}(x)}, \end{aligned}$$

after observing that the two arrays correspond to the same $d(x)$ function (the generating function of the diagonal of the matrix \mathcal{F}). In this way, the previous system allows us to obtain the functions $h(x)$ and $\bar{h}(x)$ associated to $R(x, y)$ and to $\bar{R}(x, y)$, respectively.

Theorem 6.1 *Let $\mathbf{p} = p_0 \dots p_{n-1} \in \{0, 1\}^h$ be a $=$ monotonic pattern with $|n_1^{\mathbf{p}} - n_0^{\mathbf{p}}| \in \{0, 1\}$. Then the Riordan arrays $(d(x), h(x))$ and $(d(x), \bar{h}(x))$ associated to \mathbf{p} and $\bar{\mathbf{p}}$ satisfy the following system of equations:*

$$F(x, 1) = \frac{d(x)}{1 - xh(x)} + \frac{d(x)}{1 - \bar{h}(x)} - d(x) \quad (6.2)$$

$$\bar{F}(x, 1) = \frac{d(x)}{1 - x\bar{h}(x)} + \frac{d(x)}{1 - h(x)} - d(x) \quad (6.3)$$

The previous result took for grant the possibility to compute the function $d(x)$. In fact, it is known the following relation due to Cauchy (see [16, Cap. 6, p. 182]):

$$[x^0]F(x, \frac{y}{x}) = d(y) = \frac{1}{2\pi i} \oint F(x, \frac{y}{x}) \frac{dx}{x}. \quad (6.4)$$

In order to compute the previous integral it is necessary to find the singularities $x(y)$ such that $x(y) \rightarrow 0$ with $y \rightarrow 0$ and apply the Residue theorem.

Example 6.1 *Let's study the case $\mathfrak{p} = 101$, for which we have:*

$$F(x, y) = \frac{1 + xy}{(1 - x - y)(1 + xy) + x^2y}. \quad (6.5)$$

We notice that \mathfrak{p} is a =monotonic pattern and both the arrays \mathcal{R} and $\bar{\mathcal{R}}$ are Riordan. We want to compute the integral (6.4) to find the $d(x)$ function. We have:

$$\frac{1}{x}F(x, \frac{y}{x}) = -\frac{1 + y}{-x + y - xy + x^2 + y^2},$$

and we find the good singularity

$$x_1(y) = \frac{y}{2} + \frac{1}{2} - \frac{\sqrt{1 - 2y - 3y^2}}{2}.$$

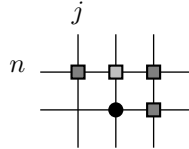
Therefore, we obtain:

$$d(t) = \lim_{x \rightarrow x_1(t)} \frac{1}{x}F(x, \frac{t}{x})(x - x_1(t)) = \frac{\sqrt{1 - 2t - 3t^2}}{1 - 3t}. \quad (6.6)$$

If we develop this function into series, we have:

$$d(t) = 1 + 2t + 4t^2 + 10t^3 + 26t^4 + 70t^5 + 192t^6 + 534t^7 + O(t^8).$$

In order to compute both $h(t)$ and $\bar{h}(t)$, we could apply Theorem 6.1 directly. However, we wish to use another approach and compute $h(t)$ by using Theorem 4.6. In fact, when $\mathfrak{p} = 101$, we have the following relation:



which translates into the generating functions $P^{[0]}(t) = 1 - t + t^2$ and $Q^{[1]}(t) = 1$. By replacing $P^{[0]}(t)$ and $Q^{[1]}(t)$ in formula (4.9) we obtain the generating function for the A -sequence

$$A(t) = P^{[0]}(t) + tA(t)Q^{[1]}(t),$$

that is,

$$A(t) = \frac{1}{1 - t} - t = 1 + t^2 + t^3 + t^4 + O(t^5), \quad (6.7)$$

thus confirming what we already noticed in the Introduction. For $h(t)$ we use formula (4.7) and obtain

$$h(t) = P^{[0]}(th(t)) + th(t)^2,$$

thus finding:

$$h(t) = \frac{1 + t - \sqrt{1 - 2t - 3t^2}}{2t(1 + t)} \quad (6.8)$$

Using formula (4.4) we can find the generating function for the Z-sequence:

$$Z(t) = \frac{2(1-t+t^2)}{1-t} = 2 + 2t^2 + 2t^3 + 2t^4 + 2t^5 + 2t^6 + 2t^7 + O(t^8). \quad (6.9)$$

We finally use equation (6.2) in Theorem 6.1 and find:

$$\bar{h}(t) = \frac{1+t-\sqrt{1-2t-3t^2}}{2t} \quad (6.10)$$

When $F(x, y)$ can be converted into partial fraction form, we can use the following substitution:

$$\begin{aligned} F(tw, \frac{1}{w}) &= R(t, w) + \bar{R}(t, \frac{1}{tw}) - d(t) \\ &= \frac{d(t)}{1-twh(t)} + \frac{d(t)}{1-\frac{1}{w}\bar{h}(t)} - d(t) \\ &= \frac{d(t)}{1-twh(t)} + d(t) \left(\frac{1}{1-\frac{1}{w}\bar{h}(t)} - 1 \right) \\ &= \frac{d(t)}{1-twh(t)} + d(t) \left(\frac{w}{w-\bar{h}(t)} - 1 \right) \\ &= \frac{d(t)}{1-twh(t)} + d(t) \frac{\bar{h}}{w-\bar{h}(t)} \\ &= \frac{d(t)}{1-twh(t)} - \frac{d(t)}{1-\frac{w}{\bar{h}(t)}}. \end{aligned} \quad (6.11)$$

We can state this result in the following theorem:

Theorem 6.2 Let $\mathbf{p} = p_0 \dots p_{h-1} \in \{0, 1\}^h$ be a monotonic pattern with $|n_1^{\mathbf{p}} - n_0^{\mathbf{p}}| \in \{0, 1\}$. Then the Riordan arrays $(d(x), h(x))$ and $(d(x), \bar{h}(x))$ associated to \mathbf{p} and $\bar{\mathbf{p}}$ satisfy the following equation:

$$F(tw, \frac{1}{w}) = \frac{d(t)}{1-twh(t)} - \frac{d(t)}{1-\frac{w}{\bar{h}(t)}}. \quad (6.13)$$

Theorem (6.2) allows to find $d(t)$, $h(t)$ and $\bar{h}(t)$ all at same time.

Example 6.2 Using the pattern $\mathbf{p} = 101$, we make the substitution (6.11) in (6.5) and obtain:

$$F(tw, \frac{1}{w}) = \frac{1+t}{(1-tw-\frac{1}{w})(1+t)+t^2w}.$$

Now, using partial fraction expansion we get:

$$F(tw, \frac{1}{w}) = \frac{(1+t)}{t(s_1-s_2)} \frac{1}{1-\frac{tw}{s_1t}} - \frac{(1+t)}{t(s_1-s_2)} \frac{1}{1-\frac{tw}{s_2t}},$$

with

$$\begin{aligned} s_1 &= \frac{1+t+\sqrt{1-2t-3t^2}}{2t} = \frac{1}{t} - t - t^2 - 2t^3 - 4t^4 - 9t^5 + O(t^6), \\ s_2 &= \frac{1+t-\sqrt{1-2t-3t^2}}{2t} = 1+t+t^2+2t^3+4t^4+9t^5+O(t^6). \end{aligned}$$

Therefore, we have:

$$\begin{aligned} d(t) &= \frac{1+t}{t(s_1-s_2)} = \frac{\sqrt{1-2t-3t^2}}{1-3t}, \\ h(t) &= \frac{1}{ts_1} = \frac{1+t-\sqrt{1-2t-3t^2}}{2t(1+t)}, \\ \bar{h}(t) &= s_2 = \frac{1+t-\sqrt{1-2t-3t^2}}{2t}. \end{aligned}$$

as expected.

7 Conclusions

The connection between the language of words avoiding a given pattern and Riordan arrays is of interest to us because the resulting matrices are better defined by means of an A -matrix rather than by an A -sequence. As we mentioned in Section 4, there is no difference between Riordan arrays defined in either way: the A -sequence is a particular case of A -matrix and, given a Riordan array defined by an A -matrix, this corresponds to a well-defined A -sequence. However, while the A -sequence is unique, the A -matrix may be not, but the main difference, that we wish to point out here, is another. Let us consider the pattern $\mathfrak{p} = 11100$ and let us apply the method described in the present paper to obtain the Riordan array $R_{n,j}$ counting the number of words of length $2n - j$ and with n bits equal to 1:

$n \setminus j$	0	1	2	3	4	5	6	...
0	1	0	0	0	0	0	0	...
1	2	1	0	0	0	0	0	...
2	6	3	1	0	0	0	0	...
3	18	9	4	1	0	0	0	...
4	58	29	13	5	1	0	0	...
5	192	96	44	18	6	1	0	...
6	650	325	151	64	24	7	1	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

The basic recurrence for the Riordan array is:

$$R_{n+1,j+1} = R_{n,j} + R_{n+1,j+2} - R_{n-2,j} \tag{7.1}$$

and this shows that the combinatorial problem is described by an A -matrix, containing simple coefficients. In principle, this should be equivalent to some combinatorial proof relating the elements of row $n + 1$, to elements in the same or in the previous rows.

The equivalence between the A -matrix and the A -sequence, however, implies a connection of the elements in row $n + 1$ from the elements of the previous rows, that is:

$$R_{n+1,j+1} = a_0 R_{n,j} + a_1 R_{n,j+1} + a_2 R_{n,j+2} + \dots$$

this relation being actually finite, since $R_{n,j} = 0$ for $j > n$. This can be important in several applications; for example, in the recent literature there exist some methods to construct the objects of a class of combinatorial structures which are mainly based on dependences of this sort (see, e.g. [1]) since they produce the combinatorial objects of size $n + 1$ starting from the objects of size n . However, if we look for the A -sequence corresponding to our simple A -matrix, we find that

$$A(t) = 1 + t + 2t^3 - t^4 + 7t^5 - 12t^6 + 38t^7 - 99t^8 + \dots$$

and this excludes that there might exist a “simple” dependence of the elements in row $n + 1$ from the elements in row n .

Obviously, our argument is not conclusive, and could exist some other connection between two consecutive rows, for example with non-constant coefficients. However, we hope that this observation might clarify the limits of some of these approaches and lead to a more general formulation of the corresponding methods.

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