# Stirling numbers for complex arguments

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**Abstract**

We define the Stirling numbers for complex values and obtain extensions of certain identities involving these numbers. We also show that the generalization is a natural one for proving unimodality and monotonicity results for these numbers. The definition is based on the Cauchy integral formula and can be used for many other combinatorial numbers.

## 1 Introduction

In this note we propose a solution to the problem of Graham, Knuth and Patashnik [3] asking for a good generalization of the Stirling numbers of the first and second kind ( $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ in a standard notation) to complex numbers $n$ and $k$. Our definition defines these numbers as a contour integral which reduces to the Cauchy integral formula when $n$ and $k$ are integers. We show that when $n-k$ is an integer some identities involving these numbers generalize nicely to the complex case while others do not with our definitions. In particular the classical recurrences involving these numbers do generalize.

The first section gives definitions and some generalized identities. Our generalization seems suited for many numbers defined as coefficients of powers of a fixed function. A counting function with $m$ parameters will become an analytic function of $m$ complex variables.

In the second section we show that these generalized functions give natural proofs of the unimodality and log-concavity of the original numbers for extensive ranges of $n$ and $k$. The difference is that we study the derivatives of the generalized functions rather than the

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differences of the original discrete functions. The definition we use is implicit in the studies of the asymptotic behaviour of various combinatorial numbers.

## 2 Definitions and Easy Consequences

We begin from the classical definitions of the Stirling numbers in terms of their generating functions:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!}[t^{n-k}]\left(\frac{1}{t}\ln\left(\frac{1}{1-t}\right)\right)^k,$$

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{n!}{k!}[t^{n-k}]\left(\frac{e^t-1}{t}\right)^k,$$

$[t^n]$ being the *coefficient of* operator. Using Cauchy's formula we have for $y, x \in \mathbf{N}$:

$$\begin{bmatrix} y \\ x \end{bmatrix} = \frac{y!}{x!}\frac{1}{2\pi i}\oint_{|z|=r} z^{-y-1}\ln^x\left(\frac{1}{1-z}\right)dz,$$

$$\left\{ \begin{matrix} y \\ x \end{matrix} \right\} = \frac{y!}{x!}\frac{1}{2\pi i}\oint_{|z|=s} z^{-y-1}\left(e^z-1\right)^x dz,$$

where $y! = \Gamma(y+1), x! = \Gamma(x+1), 0 < r < 1, 0 < s < \infty$ and the contours of integration are circles of radius $r$ and $s$ respectively. We notice however that in these formulas $x$ and $y$ can be arbitrary complex numbers ($y \notin \mathbf{Z}^-$) and so we can use them to define Stirling numbers for complex variables. We first consider the case that $x - y$ is an integer. In this case the integrands above are singlevalued so that the values of $r$ and $s$ are, subject to the constraints above, irrelevant.

**Proposition 2.1** *If $x - y \in \mathbf{N}$, then:*

$$\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} y-1 \\ x-1 \end{bmatrix} + (y-1)\begin{bmatrix} y-1 \\ x \end{bmatrix}$$

**Proof:** From definition we have:

$$\begin{bmatrix} y-1 \\ x-1 \end{bmatrix} = \frac{(y-1)!}{(x-1)!}\frac{1}{2\pi i}\oint_{|z|=r} z^{-(y-1)-1}\ln^{x-1}\left(\frac{1}{1-z}\right)dz.$$

If we integrate by parts the expression

$$\begin{bmatrix} y-1 \\ x-1 \end{bmatrix} = \frac{(y-1)!}{(x-1)!}\frac{1}{2\pi i}\oint_{|z|=r} z^{-y}\frac{(1-z)}{x}\frac{d}{dz}\left(\ln^x\left(\frac{1}{1-z}\right)\right)dz$$

we obtain:

$$\frac{(y-1)!}{(x-1)!}\left\{\frac{z^{-y}(1-z)}{2\pi i x}\ln^x\left(\frac{1}{1-z}\right)\Bigg|_{r(|z|=r)}^{r} + \right.$$

$$\left. +\frac{y}{2\pi i x}\oint z^{-y-1}\ln^x\left(\frac{1}{1-z}\right)dz - \frac{y-1}{2\pi i x}\oint z^{-y}\ln^x\left(\frac{1}{1-z}\right)dz\right\}$$

$$= \frac{y!}{x!} \oint z^{-y-1} \ln^x \left(\frac{1}{1-z}\right) dz - (y-1)\frac{(y-1)!}{x!} \oint z^{-y} \ln^x \left(\frac{1}{1-z}\right) dz$$
$$= \begin{bmatrix} y \\ x \end{bmatrix} - (y-1) \begin{bmatrix} y-1 \\ x \end{bmatrix}$$

and we have Proposition 2.1. ■

**Proposition 2.2** *If $x - y \in \mathbf{N}$, then:*

$$\left\{ \begin{matrix} y \\ x \end{matrix} \right\} = \left\{ \begin{matrix} y-1 \\ x-1 \end{matrix} \right\} + x \left\{ \begin{matrix} y-1 \\ x \end{matrix} \right\}$$

**Proof:** Since

$$\left\{ \begin{matrix} y-1 \\ x-1 \end{matrix} \right\} = \frac{(y-1)!}{(x-1)!} \frac{1}{2\pi i} \oint_{|z|=s} z^{-y}(e^z - 1)^{x-1} dz$$

and

$$x \left\{ \begin{matrix} y-1 \\ x \end{matrix} \right\} = \frac{(y-1)!}{(x-1)!} \frac{1}{2\pi i} \oint_{|z|=s} z^{-y}(e^z - 1)^{x} dz$$

we have

$$\left\{ \begin{matrix} y-1 \\ x-1 \end{matrix} \right\} + x \left\{ \begin{matrix} y-1 \\ x \end{matrix} \right\} = \frac{(y-1)!}{(x-1)!} \frac{1}{2\pi i} \oint_{|z|=s} z^{-y}(e^z - 1)^{x-1} e^z dz.$$

Integrating by parts gives:

$$= \frac{(y-1)!}{(x-1)!} \left\{ 0 - \frac{-y}{2\pi i x} \oint_{|z|=s} z^{-y-1}(e^z - 1)^x dz \right\} = \left\{ \begin{matrix} y \\ x \end{matrix} \right\},$$

hence the proposition is proved. ■

**Remarks**: The $\Gamma$ function has singularities at the negative integers. The Stirling functions do not however because the integrals in their definitions are zero when $y$ is a negative integer. The recursions in Propositions 2.1, 2.2 can define the values for $y$ a negative integer (this can also be done using a limiting argument).

We now establish a result which suggests that with suitable restrictions many classical identities generalize to complex cases. See §6.1 of Graham, Knuth and Patashnik [3] and [4, 5] by Knuth, for a fascinating survey of identities for Stirling numbers. Perhaps the most interesting one is the one below:

**Proposition 2.3** *If $x - y \in \mathbf{Z}$ then*

$$\begin{bmatrix} -y \\ -x \end{bmatrix} = \left\{ \begin{matrix} x \\ y \end{matrix} \right\}.$$

**Proof:** Set $u = 1 - e^t, du = -e^t dt$ in the definition:

$$\begin{bmatrix} y \\ x \end{bmatrix} = \frac{y!}{x!} \frac{1}{2\pi i} \oint_{C_1} u^{-y-1} \left( \ln \left( \frac{1}{1-u} \right) \right)^x du,$$

to obtain

$$= \frac{y!}{x!} \frac{1}{2\pi i} \oint_{C_2} \left( (-1) \left( e^t - 1 \right) \right)^{-y-1} (-t)^x \left( -e^t \right) dt$$

where $C_2$ is a closed path with the origin in its interior ($u = 0 \iff t = 0$). If we set $t = e^{\alpha+2\pi iz}$, $\alpha$ a real number,

$$= \frac{y!}{x!}\int_{-\frac{1}{2}}^{\frac{1}{2}}(-1)^{x-y}\left(e^{e^{\alpha+2\pi iz}}-1\right)^{-y-1}\left(e^{\alpha+2\pi iz}\right)^{x+1}e^{e^{\alpha+2\pi iz}}dz.$$

Here we observe that $(-1)^a$ is, unless $a$ is an integer, a multivalued function. We have not derived an identity for general $a$ so not for general $x$ and $y$. With our hypothesis we have:

$$= \frac{y!}{x!}(-1)^{x-y}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(e^{e^{\alpha+2\pi iz}}-1\right)^{-y-1}\left(e^{\alpha+2\pi iz}\right)^{x+1}\left(e^{e^{\alpha+2\pi iz}}-1+1\right)dz$$

$$= \frac{y!}{x!}(-1)^{x-y}\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(e^{e^{\alpha+2\pi iz}}-1\right)^{-y}\left(e^{\alpha+2\pi iz}\right)^{x+1}dz+\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(e^{e^{\alpha+2\pi iz}}-1\right)^{-y-1}\left(e^{\alpha+2\pi iz}\right)^{x+1}\right\}.$$

Then

$$\begin{bmatrix}-y\\-x\end{bmatrix} = \frac{(-y)!}{(-x)!}(-1)^{y-x}\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(e^{e^{\alpha+2\pi iz}}-1\right)^{y}\left(e^{\alpha+2\pi iz}\right)^{-x+1}dz+\right.$$

$$\left.\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(e^{e^{\alpha+2\pi iz}}-1\right)^{y-1}\left(e^{\alpha+2\pi iz}\right)^{-x+1}\right\}$$

and using the identity $(-x)! = 1/((x-1)!\sin(\pi x))$,

$$= \frac{\sin(\pi x)}{\sin(\pi y)}(-1)^{y-x}\left(y\left\{\begin{matrix}x-1\\y\end{matrix}\right\}+\left\{\begin{matrix}x-1\\y-1\end{matrix}\right\}\right).$$

If $x-y$ is an integer then $\sin(\pi x)/\sin(\pi y) = (-1)^{x-y}$ and by applying Proposition 2.2 we conclude our proof. ■

If $x-y$ is not an integer, since the integrand is not singlevalued, the values of $r$ and $s$ become important. We choose to define $r$ and $s$ by saddlepoint conditions, that is by $1/((1-r)\ln(1/(1-r))) = y/x$ and $s\exp(s)/(\exp(s)-1) = y/x$. We then choose the contour $z = s\exp(i\theta)$ or $z = r\exp(i\theta)$ and integrate from $\theta = -\pi$ to $\theta = \pi$. (When $x$ and $y$ are real this ensures that, as will be seen, the asymptotic behaviour of the Stirling numbers can be obtained from the formulas for integer $n$ and $k$ by replacing $n$ and $k$ by $y$ and $x$.) When $x$ and $y$ are complex we propose again defining $r$ and $s$ by the same equations. (This defines them as analytic functions of $x$ and $y$ by the implicit function theorem since the derivatives of the left hand side with respect to $r$ or $s$ is not 0 for $r$ and $s$ sufficiently close to the positive real axes (thus for $x$ and $y$ sufficiently close to the real axes)). We now choose the contours $z = r\exp(i\theta)$ or $z = s\exp(i\theta)$ where again $\theta$ goes from $-\pi$ to $\pi$. Note for $r$ and $s$ small we can expand the integrand as a power series in $z$ and integrate term by term. This gives a uniformly convergent series of analytic functions of $x$ and $y$. Note that when $x$ and $y$ are positive integers we get the standard Stirling numbers. The analytic function can be analytically continued.

**Remark**: It seems to us that the ideas used to derive identities and recurrences when $x$ and $y$ differ by an integer lead to complicated formulas in general. There are significant terms resulting from the fact that the integrands are not singlevalued and also from the fact that the contours change with $x$ and $y$.

We found that the contour integrals in the above definitions can be evaluated readily using Maple (see [1]) if they are written as an integral over $z$ from $-1/2$ to $1/2$ as was done in the proof of Proposition 2.3. All the propositions above were checked for several values of $x$ and $y$. For example, when $y = 7.675$ and $x = 3.675$ we have $\left\{ {y \atop x} \right\} \approx 1011.174104$ and $\left[ {-x \atop -y} \right] \approx 1011.174040$. Finally note that Sprugnoli and Del Lungo [9] have considered the problem of generalizing the identity

$$\sum_k \binom{n}{k} \left\{ {k \atop m} \right\} = \left\{ {n+1 \atop m+1} \right\}$$

to real $n$ and $m$. They show that such a generalization would only be possible for $|m| < 1$ using asymptotic estimates. We had difficulty evaluating the relevant integrals and so could not test the identity. Anyway, our definition gives the same asymptotic behaviour as they got, so with our definition also we need $-1 < m < 1$ for the identity to be true. We do not continue to prove generalized recurrences and identities for other combinatorial counting functions here, rather we give examples showing that they generalize results concerning unimodality and logconcavity. They also simplify the analytic proofs of such results.

# 3 Logconcavity and unimodality results

The definitions in Section 2 are implicit in the asymptotic analysis of many combinatorial numbers since the Cauchy integral formula is often used. The analytical approach to proving that $a_{n,k}$ is unimodal, i.e. that

$$a_{n,1} \leq a_{n,2} \leq \cdots a_{n,k} \geq a_{n,k+1} \geq \cdots \geq a_{n,n},$$

or to proving that $a_{n,k}$ is logconcave, i.e. that

$$a_{n,k+1} a_{n,k-1} \geq a_{n,k}^2,$$

is to study the asymptotic behaviour of the first difference of $a_{n,k}$ or the second difference of $\ln a_{n,k}$. When the saddle point method is used to do this the contour chosen depends upon $n$ and $k$ so when $k$ changes so does the contour. The change in contour usually is not important but it is necessary to prove this. If $k$ is a real variable we can study the derivative of $a_{n,k}$ which gives us a different point of view and as we shall see the same contour may be used for all the derivatives.

We illustrate this with the entries in convolution matrices, where

$$a_{n,k} = \frac{n!}{k!}[t^n]h(t)^k.$$

We have seen that the Stirling numbers of both kinds are of this form. The asymptotic behaviour of such $a_{n,k}$ has been studied by many authors. We shall rely heavily upon the paper by Daniele Gardy [2] which includes an excellent survey of the results so far. Gardy [2] defines

$$\Delta f(z) = z\frac{d}{dz}\ln f(z) = z\frac{f'(z)}{f(z)}, \qquad \delta f(z) = \frac{f''(z)}{f(z)} - \left(\frac{f'(z)}{f(z)}\right)^2 + \frac{f'(z)}{zf(z)}$$

and suppose that $f$ satisfies the following properties ($f(z) = f_0 + f_1 z + f_2 z^2 + \cdots$)

**Assumption 3.1** *The function $f$ has real positive coefficients with $f_0 \neq 0$ and $f_1 \neq 0$, and a strictly positive, possibly infinite radius of convergence $R$.*

She also suppose that $\Psi$ satisfies the following property (her assumption is more general than that below but we do not need the more general form):

**Assumption 3.2** *The function $\Psi$ has positive coefficients such that $\Psi(0) = 0$, and has a strictly positive radius of convergence.*

The following theorems of Gardy [2] will be very usefull to us (Theorems 8, 5 and 6 of [2]):

**Theorem 3.3** *Let $f$ satisfy Assumption 3.1, and let $\Psi$ satisfy Assumption 3.2. Assume that the equation $\Delta f(z) = n/d$ has a real positive solution $\rho$ smaller than the radius of convergence of $f$ and of $\Psi$. Then, for $n, d \to \infty$ and $\eta \leq n/d \leq M$ where $\eta$ and $M$ are positive constants,*

$$[z^n]\left\{f^d(z)\Psi(z)\right\} = \frac{f^d(\rho)\Psi(\rho)}{\rho^{n+1}\sqrt{2\pi d\delta f(\rho)}}(1 + o(1))$$

■

**Theorem 3.4** *Let $f$ be a function satisfying the Assumption 3.1 and such that*

$$f(z) = e^{P(z)}$$

*where $P(z) = \sum_{0 \leq j \leq q} p_j z^j$ is a polynomail of degree $q > 1$ with positive coefficients. Let $n, d \to \infty$ in such a way that $d = o(n)$ but $(\ln n)^{3q} n^{2q-3} = o\left(d^{2(q-1)}\right)$, and define $\rho$ as the unique real positive solution of $\rho P'(\rho) = n/d$. Then*

$$[z^n]e^{dP(z)} = \frac{e^{dP(\rho)}}{\rho^n\sqrt{2\pi n}}(1 + o(1)).$$

■

**Theorem 3.5** *Let $f$ be a meromorphic function with positive coefficients, whose singularity of smallest modulus is a pole at 1 of order $p$: $f(z) = g(z)/(1-z)^p$, where $g$ is a function analytic for $|z| \leq 1$ and with positive coefficients. Assume that $f_1 \neq 0$ and define $\rho$ by $\Delta f(\rho) = n/d$. Then if $d = o(n)$ and $\ln(n/\sqrt{d}) = o(d^{1/3})$ we have*

$$[z^n]f^d(z) = \sqrt{\frac{\rho d}{2\pi}} \frac{f^d(\rho)}{n\rho^n} (1 + o(1)).$$

■

We shall make some minor changes to these three theorems for our purposes. Note that these theorems are proved by the saddle point method and are true when $n$ and $d$ are real numbers with our definition of $[z^n]f^d(z)$.

Suppose $a_{y,x}$ is defined by

$$a_{y,x} = \frac{y!}{x!}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-y(\alpha+i\theta)+x\ln h(e^{\alpha+i\theta})}d\theta = \frac{y!}{x!}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x, y, \theta)d\theta$$

where $\Delta h(e^{\alpha}) = y/x$. Then:

$$\frac{da_{y,x}}{dx} = y!\frac{-(x!)'}{(x!)^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y,\theta)d\theta$$

$$+\frac{y!}{x!}\int_{-\pi}^{\pi} f(x,y,\theta)\left(-y\frac{d\alpha}{dx} + \ln h(e^{\alpha+i\theta}) + x\frac{h'(e^{\alpha+i\theta})}{h(e^{\alpha+i\theta})}e^{\alpha+i\theta}\frac{d\alpha}{dx}\right)d\theta.$$

If we consider the proof of Theorem 3.3 (8 of [2]) we see that one difference is the factor $\ln h(\rho e^{i\theta})$ in the integrand, where $\rho = e^{\alpha}$. Gardy shows that $\delta h(\rho) \sim cy$ and

$$\ln h(\rho e^{i\theta}) = \ln h(\rho) + i\theta\Delta h(\rho) - \theta^2\delta h(\rho) + \cdots$$

Gardy shows that the integral over $\theta = -\alpha$ to $\theta = \alpha$ where $\alpha = \ln y/\sqrt{y}$ gives the asymptotic behaviour of the whole integral. Note however that the coefficient of $\theta^2$ in $\ln h(\rho e^{i\theta})$ is $1/x$ times that in $x \ln h(\rho e^{i\theta})$. Thus with this choice of $\alpha$ we have $\delta h(\rho)\alpha^2 = o(1)$ ( and the terms involving higher powers of $\theta$ are even smaller. ( Gardy shows that all the coefficients of the powers of $\theta$ are the same size). Now

$$\int_{-\alpha}^{\alpha} e^{-x\rho\ln h(\rho)\theta^2}\theta d\theta = 0.$$

Thus

$$\frac{y!}{x!}\int_{-\pi}^{\pi} f(x,y,\theta)\ln h(e^{\alpha+i\theta})d\theta = \ln h(\rho)\frac{y!}{x!}\int_{-\pi}^{\pi} f(x,y,\theta)d\theta\left(1+O\left(\frac{\ln^2 y}{y}\right)\right)$$

$$= a_{y,x}\ln h(\rho)\left(1+O\left(\frac{\ln^2 y}{y}\right)\right).$$

Since $|h(\rho e^{i\theta})| \leq |h(\rho)|$ by Assumption 3.2 and since $h(\rho e^{i\theta})^x \ln h(\rho e^{i\theta}) = 0$ for $x > 0$, if $h(\rho e^{i\theta}) = 0$ then the $\ln h(e^{\alpha+i\theta})$ term is unimportant for Gardy's analysis for $|\theta| > |\alpha|$, hence this range of $\theta$ is negligable. Similar considerations apply to

$$\frac{h'(\rho e^{i\theta})}{h(\rho e^{i\theta})}\frac{d\rho}{dx}e^{i\theta} = \frac{d\rho}{dx}\left(\frac{h'(\rho)}{h(\rho)} + i\theta\rho\ln(\rho) - \theta^2\left.\frac{d^3\ln h(\rho e^{i\theta})}{d^3\theta}\right|_{\theta=0} + \cdots\right)e^{i\theta}.$$

Furthermore

$$\int_{-\pi}^{\pi} f(x,y,\theta)\frac{h'(e^{\alpha+i\theta})}{h(e^{\alpha}+i\theta)}e^{\alpha+i\theta}\frac{d\alpha}{dx}d\theta = \int_{-\pi}^{\pi} f(x,y,\theta)\frac{h'(e^{\alpha})}{h(e^{\alpha})}e^{\alpha}\frac{d\alpha}{dx}d\theta\left(1+O\left(\frac{\ln^2 y}{y}\right)\right).$$

Since if $\rho = e^{\alpha}$ then $d\alpha/dx = \rho^{-1}d\rho/dx$. Thus:

$$-y\frac{d\alpha}{dx} + x\frac{h'(e^{\alpha})}{h(e^{\alpha})}e^{\alpha}\frac{d\alpha}{dx} = \left(\frac{-y}{\rho} + x\frac{h'(\rho)}{h(\rho)}\right)\frac{d\rho}{dx} = 0$$

since $\Delta h(e^{\alpha}) = y/x$. Thus

$$\frac{da_{y,x}}{dx} = a_{y,x}\left(\frac{d}{dx}(-\ln\Gamma(x+1)) + \ln(\rho)\right)\left(1+O\left(\frac{\ln^2 y}{y}\right)\right).$$

Moreover

$$\frac{d^2 a_{y,x}}{d^2 x} = \frac{y!}{x!}\frac{d^2\left(-\ln\Gamma(x+1)\right)}{d^2 x}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y,\theta)d\theta$$
$$+2\frac{y!}{x!}\frac{d(-\ln\Gamma(x+1))}{dx}\frac{1}{2\pi}\frac{d}{dx}\int_{-\pi}^{\pi} f(x,y,\theta)d\theta$$
$$+\frac{y!}{x!}\frac{1}{2\pi}\frac{d^2}{d^2 x}\int_{-\pi}^{\pi} f(x,y,\theta)d\theta.$$

We again have that we can evaluate all the derivatives with respect to $x$ at $\theta = 0$. We also use that the expression $-y/\rho + (xh'(\rho)/h(\rho)(d\rho/dx)$ and its derivative equal 0. Hence:

$$\frac{d^2 a_{y,x}}{d^2 x} = a_{y,x}\left(-\frac{1}{x} - 2\ln xh(\rho) + \left(\ln^2 h(\rho) + \frac{h'(\rho)}{h(\rho)}\frac{d\rho}{dx}\right)\right)\left(1 + O\left(\frac{\ln^2 y}{y}\right)\right).$$

Note also

$$\frac{d\ln a_{y,x}}{dx} = \frac{(a_{y,x})'}{a_{y,x}}, \qquad \frac{d^2\ln a_{y,x}}{d^2 x} = \frac{(a_{y,x})''}{a_{y,x}} - \left(\frac{(a_{y,x})'}{a_{y,x}}\right)^2.$$

Using the fact that the asymptotic expansion for $\ln\Gamma(x)$ may be differentiated term by term, we find:

**Theorem 3.6** *Under the assumptions of Theorem 3.3 (see above)*

$$\frac{da_{y,x}}{dx} \sim a_{y,x}\left(\ln h(\rho) - \ln x\right),$$

$$\frac{d^2\ln a_{y,x}}{d^2 x} \sim a_{y,x}\left(-\ln^2 x - \frac{1}{x} + \frac{h'(\rho)}{h(\rho)}\frac{d\rho}{dx}\right)$$

■

Let us now consider the proof of Theorem 3.4 (5 of [2]). The arguments concerning $\ln h(\rho e^{i\theta})$ and $h'(\rho e^{i\theta})/h(\rho e^{i\theta})$ in the proof of Theorem 3.6 are valid. With Gardy's choice of $\alpha$ we have $\alpha^2 x\delta h(\rho) \to \infty$ but $\alpha^2\delta h(\rho) \to 0$. The analysis of Gardy is easily modified to handle the case $q = 1$, we found that $\alpha = \sqrt{\ln y}/y$ works if $q = 1$. We conclude therefore:

**Theorem 3.7** *The conclusions of Theorem 3.6 hold when $h(z) = \exp(P(z))$ where $P(z)$ is a polynomial of degree $q \geq 1$ with positive coefficients provided $x, y \to \infty$ in such a way that $x > y^\epsilon$, $\epsilon$ a positive constant, if $q = 1$ and $x > y^{a+\epsilon}$, $a = (2q-3)/(3(q-1))$ if $q \geq 2$.*

■

We now consider the proof of Theorem 3.5 (6 of [2]). First of all when $p \geq 1$ the singularity may be of the form $g(z)\ln(1/(1-z))/(1-z)^p$ since $\ln(1/(1-z))$ is a slowly-varying function. Furthermore since $d\ln(1/(1-z))/dz = (1-z)^{-1}$ the analysis of Gardy is easily modified to handle the case when the singularity is of the form $g(z)\ln(1/(1-z))$. The derivatives of $\ln\ln(1/(1-z))$ are much like those for a singularity $1/(1-z)$, there are various powers of $\ln(1/(1-z))$ which do not matter. Also $|\ln(1/(1-\rho e^{i\theta}))| \leq |\ln(1/(1-\rho e^{i\alpha}))|$ for $\theta \geq \alpha$ (this seems to be well known). The terms $\ln h(\rho e^{i\theta})$ and $h'(\rho e^{i\theta})/h(\rho e^{i\theta})$ may be handled as above, so we conclude:

**Theorem 3.8** *The conclusions of Theorem 3.6 hold when $h$ is a meromorphic function with positive coefficients, whose singularity of smallest modulus is at $r$ and is of the form: $g(z)\ln(1/(r-z))/(r-z)^p$ where $p \geq 0$ or of the form $g(z)/(z-r)^p$, where $p \geq 1$. Here $g(z)$ is a function analytic for $z \leq r$ and with positive coefficients. Assume $[z]h(z) \neq 0$ and define $\rho$ by $\Delta f(\rho) = y/x$. Then if $x = o(y)$ but $x \geq y^\epsilon$, $\epsilon$ a constant, the conclusion of Theorem 3.6 hold.*

■

With all of Theorems 3.6, 3.7 and 3.8 the following corollary is useful:

**Corollary 3.9** *If*

$$\frac{1}{\rho} + \frac{h''(\rho)}{h'(\rho)} - \frac{h'(\rho)}{h(\rho)} > 0,$$

*then $d\rho/dx < 0$ and since $h'(\rho) > 0$ it follows that $a_{y,x}$ is log-concave (hence unimodal).*

**Proof:** From the saddle point condition

$$\rho\frac{h'(\rho)}{h(\rho)} = \frac{y}{x} \qquad \text{or} \qquad \ln\rho + \ln h'(\rho) - \ln h(\rho) = \ln y - \ln x.$$

Hence

$$\left(\frac{1}{\rho} + \frac{h''(\rho)}{h'(\rho)} - \frac{h'(\rho)}{h(\rho)}\right)\frac{d\rho}{dx} = \frac{-1}{x}.$$

■

# 4 Applications

In the applications we shall use the fact that $a_{y-x,x}$ defined with a certain $f(z)$ ($h(z)$ in our definition) is equal to $a_{y,x}$ defined with $h(z) = zf(z)$. Since the integrand is the same in the both cases we shall apply Theorems 3.6, 3.7 or 3.8 with $h(z)/z$ so that Assumption 3.1 holds but use $h(z)$ in Corollary 3.9. We consider some examples of Merlini, Sprugnoli and Verri [8]. Our logconcavity results hold for every $y$ sufficiently large of course.

## 4.1 Stirling numbers of the second kind

Set $h(z) = \exp(z) - 1$, then

$$\frac{h''(\rho)}{h'(\rho)} = 1, \qquad \frac{h'(\rho)}{h(\rho)} = \frac{e^\rho}{e^\rho - 1}.$$

Thus

$$\frac{1}{\rho} + 1 - \frac{e^\rho}{e^\rho - 1} = \frac{1}{\rho} - \frac{1}{e^\rho - 1} = \frac{1}{\rho} - \frac{1}{\rho + \frac{\rho^2}{2}\cdots} > 0.$$

Thus the Stirling numbers of the second kind are log-concave for $y^\epsilon \leq x \leq y$ since the conditions of Theorems 3.6 and 3.7 are satisfied. ( Also Corollary 3.9 holds.) Note also the maximum is achieved at $x_0$ where

$$\ln(e^\rho - 1) = \ln x_0 \qquad \text{or} \qquad e^\rho = x_0 + 1.$$

Also since $y/x_0 = \Delta \ln(e^\rho - 1) = \rho e^\rho/(e^\rho - 1)$ we have

$$\frac{y}{x_0} = \rho \frac{x_0+1}{x_0} = \ln(x_0+1)\frac{x_0+1}{x_0}.$$

Thus $x_0 \sim y/\ln y$, a well-known result of course.

The same analysis holds for $h(z) = \exp(P(z))$ of course and identifies the maximum. For the Stirling numbers it is easy to see that $\left\{ {y \atop x} \right\} = y^x/\Gamma(x+1)(1+o(y^{-\delta})$ if $x = O(y^\epsilon), y \to \infty, x \geq 1$, so we can conclude that $\left\{ {y \atop x} \right\}$ is logconcave for $1 \leq x \leq y$.

## 4.2 Stirling numbers of first kind

If $h(z) = \ln(1/(1-z))$ then

$$\frac{h''(\rho)}{h'(\rho)} - \frac{h'(\rho)}{h(\rho)} = \frac{1}{1-\rho} - \frac{1}{(1-\rho)\ln\left(\frac{1}{1-\rho}\right)} > 0,$$

so $\left[ {y \atop x} \right]$ is logconcave for $y^\epsilon \leq x \leq y$ since Theorem 3.8 applies. We see however that the maximum would be at $\ln(1/(1-\rho)) = x$ so $\rho = 1 - 1/\rho + o(1/\rho^2)$ hence $x \sim \ln y$. This is correct but all we have proved is that $\left[ {y \atop x} \right]$ is logconcave and monotone decreasing for $y^\epsilon \leq x \leq y$ using Theorem 3.8 and Corollary 3.9.

## 4.3 Tree polynomials

If $h(z) = \ln(1/(1-T(z)))$, where $T(z)$ is the tree function defined by $T(z) = z\exp(T(z))$. We are now studying the tree polynomials of Knuth and B. Pittel [6]. It's not hard to see that Theorem 3.8 and Corollary 3.9, apply. Furthermore

$$\frac{h'(\rho)}{h(\rho)} = \frac{T'(\rho)}{\ln\left(\frac{1}{1-T(\rho)}\right)(1-T(\rho))}, \qquad \frac{h''(\rho)}{h'(\rho)} = \frac{T''(\rho)}{T'(\rho)} + \frac{T'(\rho)}{1-T(\rho)},$$

so

$$\frac{h''(\rho)}{h'(\rho)} - \frac{h'(\rho)}{h'(\rho)} > 0.$$

Thus the tree-polynomials are logconcave and monotone decreasing for $y^\epsilon < x < y$. (The results of Meir and Moon [7] apply to the case $x < y^\epsilon$ for $x$ and $y$ integers so hopefully one can prove that the Knuth-Pittel tree polynomials are logconcave by proving their results for real $x$ and $y$.)

**Remark:** The range $x > y^\epsilon$ can be replaced by $x > (\ln y)^M$ in our Theorems 3.7 and 3.8 if $q = 1$ (and Gardy specifies the $M$ in the latter case). The saddle point method does not deal well with $x = O(\ln y)^M$. It would be useful to extend the range of Theorems 3.7 and 3.8 to this range of $x$ so that the logconcavity results hold for all interesting real values of $x$ and $y$. This would seem to be feasible since it amounts to extending the results for finite $x$ (provable by standard methods) to $x = O(\ln y)^M$ and $y \to \infty$.

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