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On some tiling games

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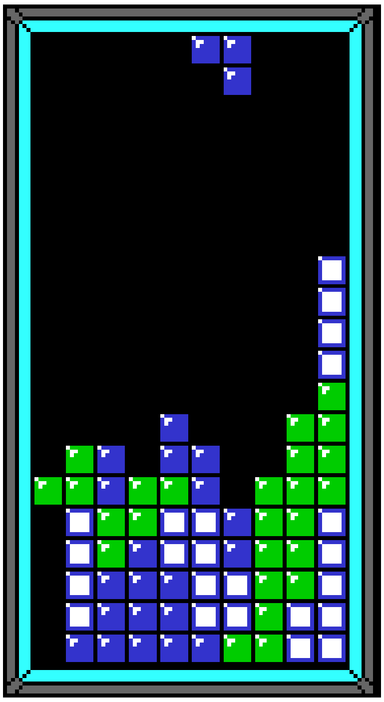
The original idea: the tetris game

Tetris is a computer game which has obsessed many computer users and attracted much attention, despite the simplicity of its rules.

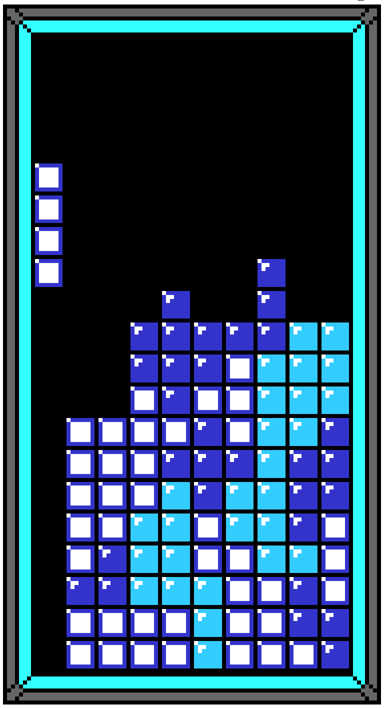
The game takes place on a grid 10 units wide and 20 units tall. When the game starts, the board is empty. Then tetrominoes, groups of four connected cells, each cell covering exactly one grid square, appear at the top of the board and fall row by row toward the bottom of the board.



When a tetromino reaches the bottom or a point where it can fall no further it remains in that spot and another tetromino appears at the top of the board.

The player uses rotations and horizontal translations to orient the tetrominoes as they fall, attempting to cover rows of the board with cells.

When a row is covered, the cells on that row are removed from the board and the cells of the rows above drop down to fill gap.



If the player does not fill the rows fast enough, eventually there will be no room on the board to place tetrominoes and the game will end.

A model for tetris games

We have a grid of dimension $p \times q$ and a set of pieces (not necessarily tetrominoes) which are uniformly generated at random.

The pieces can fall from any position or (another game) the pieces fall from the same position.

The player can translate a piece and eventually rotate it (two different games).

We get score as soon as we completely fill a line (which in this case is removed).

When there is no more room on the board to place pieces the game ends.

Suppose the player can make a mistake with a given probability: which is the average score (average number of full lines) he can obtain?

By a mistake we mean a move, which doesn't increase the score and/or leads to a *difficult* configuration of the grid (the concept of difficult configuration is considered case for case).

Some Bibliography

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The p strip tiling problem

In [7], the authors study the general puzzle-problem of counting the number of different ways a $p \times n$ strip ($p \in \mathbf{N}$ fixed, $n \in \mathbf{N}$) can be tiled with some sort of *pieces*.

The problem is approached by proving the following basic results:

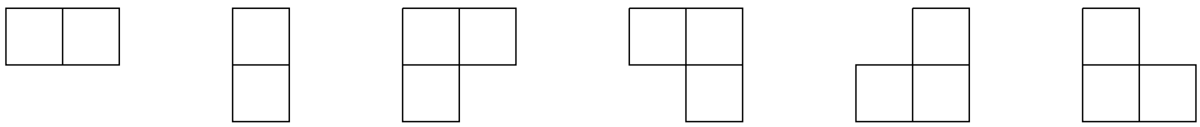
1) every puzzle-problem is equivalent to a regular grammar (i.e., the set of tilings is a regular language);

2) an algorithm exists that can find the regular grammar corresponding to a puzzle-problem;

3) as a consequence of 1) and 2), it is possible to find the rational function that counts the tilings in terms of the length of the strip and/or other parameters.

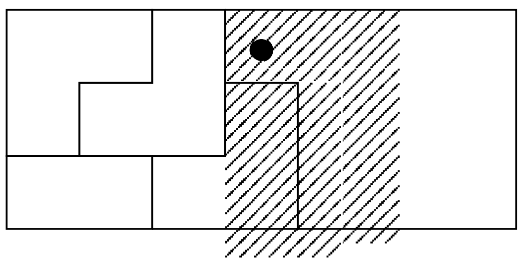
The idea of *state*

Let $\mathcal{P} = \{P_1, P_2, \cdots, P_s\}$ be a set of oriented pieces and $r = \max\{\text{length}(P_i) | P_i \in \mathcal{P}\}$.



A *state* is a $p \times r$ strip whose cells can be either *occupied* or *free*

Let us consider $p = 3$ and the previous oriented pieces:

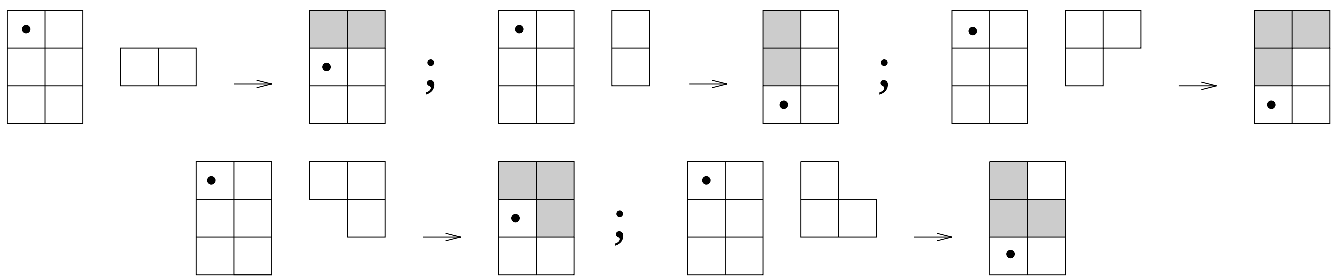


The marked cell is the *pivot cell* and we can always assume that the new piece is added in such a way that it covers the pivot cell.

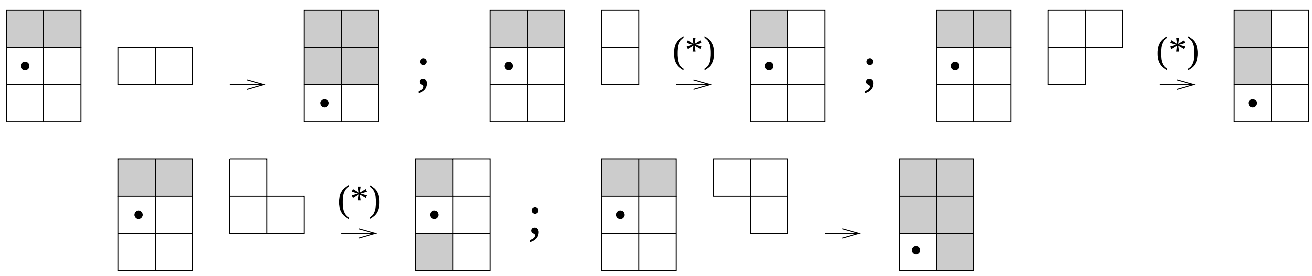
The added piece cannot extend more than r positions to the right and the $p \times r$ substrip containing the pivot cell in its leftmost column is the only part of the strip affected by the insertion of the new piece. This is our concept of state .

The initial state is the state of the strip at the beginning of the tiling process and so it is a $p \times r$ strip containing only free cells.

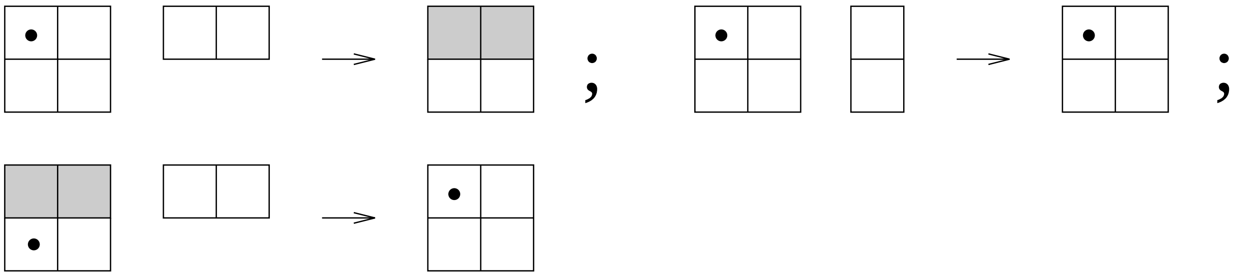
We procede by adding a piece to a state so that it covers the pivot cell:



and eventually delete the completely occupied leftmost columns (if any) and add an equal number of free cell columns to the right:



Dimers on $2 \times n$



$$T ::= \epsilon \mid Tv \mid Ah$$

$$A ::= Th$$

By using Schützenberger's methodology we now find the generating function $T(t)$ for the number of tilings of length n : we get the system

$$T(t) = 1 + tT(t) + tA(t)$$

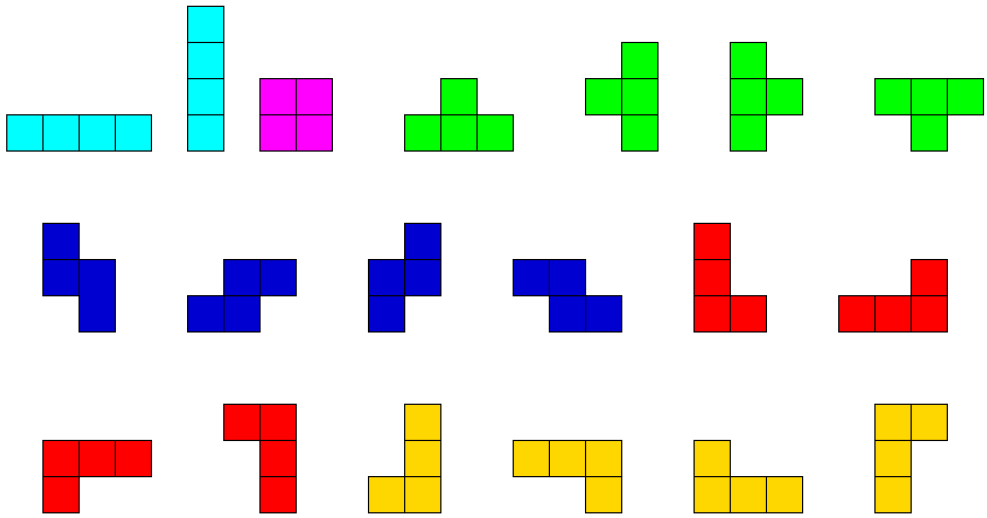
$$A(t) = tT(t)$$

whose solution is the displaced Fibonacci function:

$$T(t) = \frac{1}{1 - t - t^2} = \frac{1}{t}F(t).$$

Tetrominoes on $4 \times n$

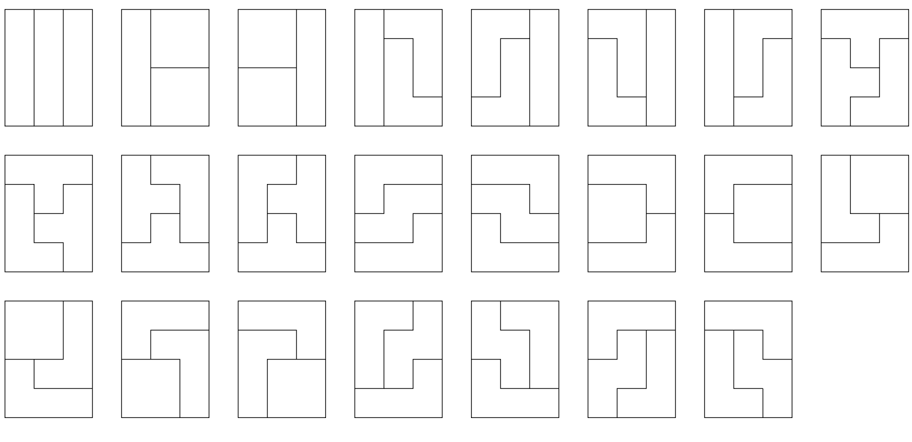
With the following 19 oriented pieces



we obtain:

$$T(t) = \frac{1 - t - 6t^2 - t^3 + \ldots + 3t^{30} + t^{31}}{1 - 2t - 8t^2 + \ldots - 3t^{34} - t^{35}} =$$

$$= 1 + t + 4\,t^2 + 23\,t^3 + 117\,t^4 + 454\,t^5 + O\left(t^6\right)$$



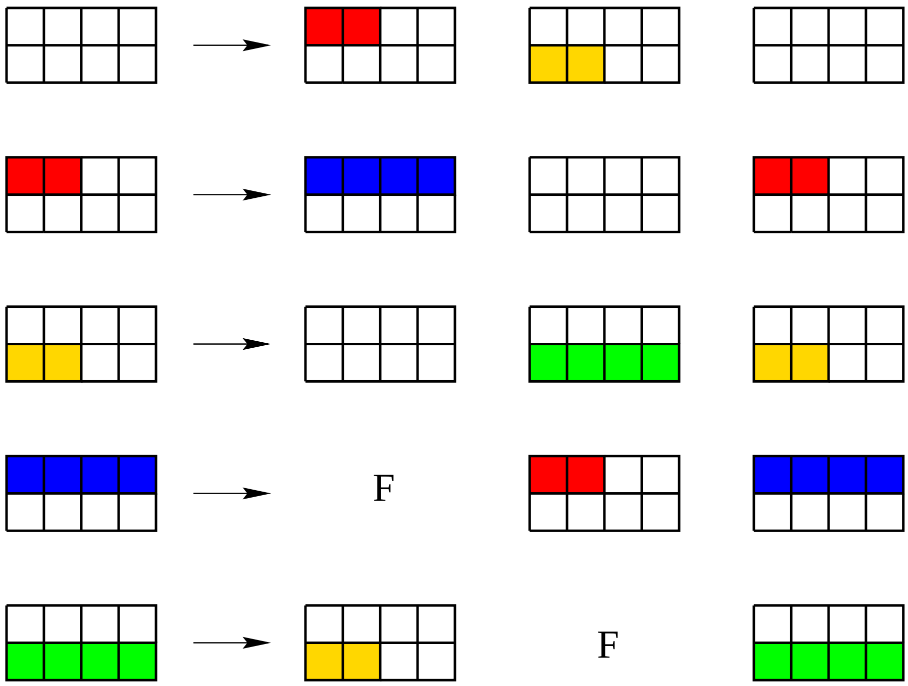
The new algorithm

Given a set of pieces $\mathcal{P}$ and a grid $p \times q$:

- A state is a $p \times q$ grid whose cells can be either occupied or free;
- the initial state is a $p \times q$ grid containing only free cells;
- we proceed by adding an oriented piece to a state in any position;
- we delete the completely occupied columns (in all positions) and add an equal number of free columns to the right.

What we get is a finite state automaton (or, equivalently, e regular grammar) to which we can apply Schützenberger's methodology. We can now assign probabilities to the possible moves and we can image to play various games by varying the values of these probabilities.

Dimers on the grid 2×4 : game 1



$$T = \frac{1}{3} tA + \frac{1}{3} tB + \frac{1}{3} twT$$

$$A = tpC + Tt\left(\frac{2}{3} - p\right) w^2 + \frac{1}{3} twA$$

$$B = Tt\left(\frac{2}{3} - p\right) w^2 + tpD + \frac{1}{3} twB$$

$$C = tp + t\left(\frac{2}{3} - p\right) w^2 A + \frac{1}{3} twC$$

$$D = t\left(\frac{2}{3} - p\right) w^2 B + tp + \frac{1}{3} twD$$

Game 1

The indeterminate t counts the number of pieces while the indeterminate w counts the score, that is the number of full lines:

$$T(t,w) = \frac{6p^2t^3}{(3-tw)\left(3\,t^2p^2w^2 - t^2w^2 - 2\,tw + 3\right)}$$

Average score:

$$M_w = \left.\frac{\partial T(t,w)}{\partial w}\right|_{t,w=1} = \frac{8-9\,p^2}{6p^2}$$

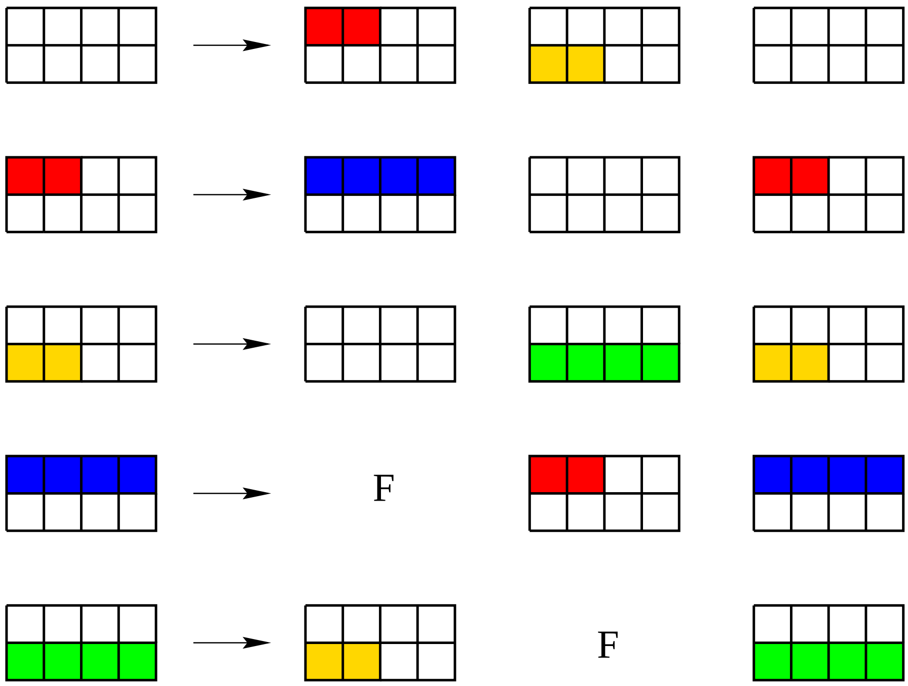
Average number of pieces:

$$M_t = \left.\frac{\partial T(t,w)}{\partial t}\right|_{t,w=1} = \frac{8+9\,p^2}{6p^2} = 3 + M_w$$

Variance:

$$V_w = V_t = \frac{27\,p^4 - 120\,p^2 + 64}{36p^4}$$

Dimers on the grid 2×4 : game 2



$$T = \frac{1}{4} tA + \frac{1}{4} tB + \frac{1}{2} twT$$

$$A = tpC + Tt\left(\frac{1}{2} - p\right) w^2 + \frac{1}{2} twA$$

$$B = Tt\left(\frac{1}{2} - p\right) w^2 + tpD + \frac{1}{2} twB$$

$$C = tp + t\left(\frac{1}{2} - p\right) w^2 A + \frac{1}{2} twC$$

$$D = t\left(\frac{1}{2} - p\right) w^2 B + tp + \frac{1}{2} twD$$

Game 2

In this case we obtain:

$$T(t,w) = \frac{-t^3p^2}{t^3w^3p^2 - 2\,t^2w^2p^2 - t^2w^2 + 3\,tw - 2}$$

Average score:

$$M_w = \left.\frac{\partial T(t,w)}{\partial w}\right|_{t,w=1} = \frac{1-p^2}{p^2}$$

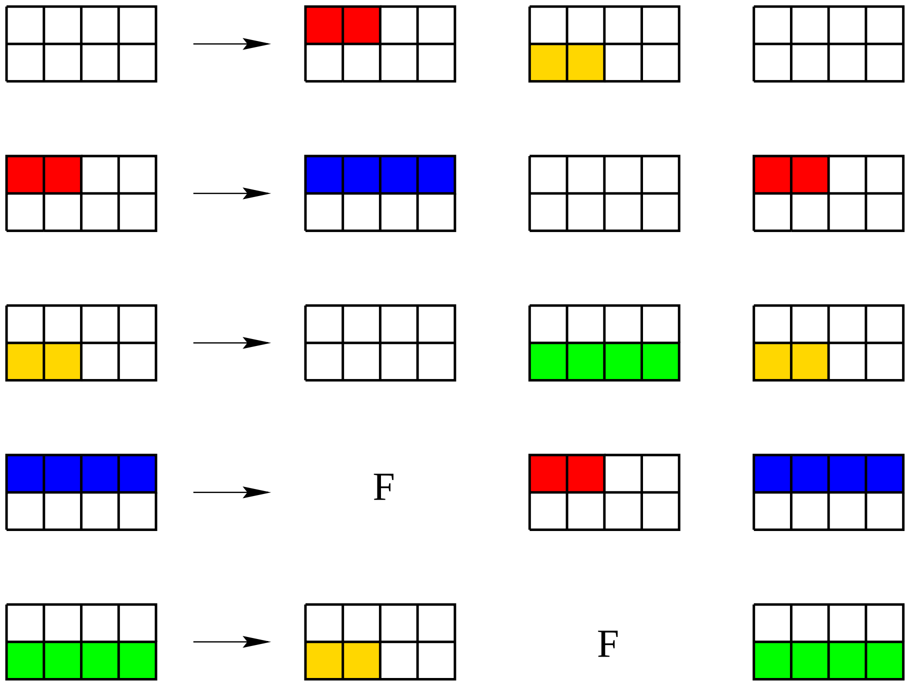
Average number of pieces:

$$M_t = \left.\frac{\partial T(t,w)}{\partial t}\right|_{t,w=1} = \frac{2p^2+1}{p^2} = 3 + M_w$$

Variance:

$$V_w = V_t = \frac{2\,p^4 - 3\,p^2 + 1}{p^4}$$

Dimers on the grid 2×4 : game 3



$$T = p\,tA + p\,tB + (1-2p)\ twT$$

$$A = tpC + \frac{2}{3}(1-p)\,tw^2T + \frac{1}{3}(1-p)\ twA$$

$$B = \frac{2}{3}(1-p)\,tw^2T + tpD + \frac{1}{3}(1-p)\ twB$$

$$C = tp + \frac{2}{3}(1-p)\,tw^2A + \frac{1}{3}(1-p)\ twC$$

$$D = \frac{2}{3}(1-p)\,tw^2B + tp + \frac{1}{3}(1-p)\ twD$$

Game 3

In this case we obtain:

$$T(t,w) = 2\,p^3 t^3 + \frac{2}{3}\,p^3 w\,(5 - 8\,p)\,t^4 +$$

$$+\frac{2}{3}\,p^3 w^2\left(6 + 11\,p^2 - 14\,p\right)t^5 + O\left(t^6\right)$$

Average score:

$$M_w = \frac{-54\,p^3 + 31\,p^2 + 10\,p + 4}{18 p^3}$$

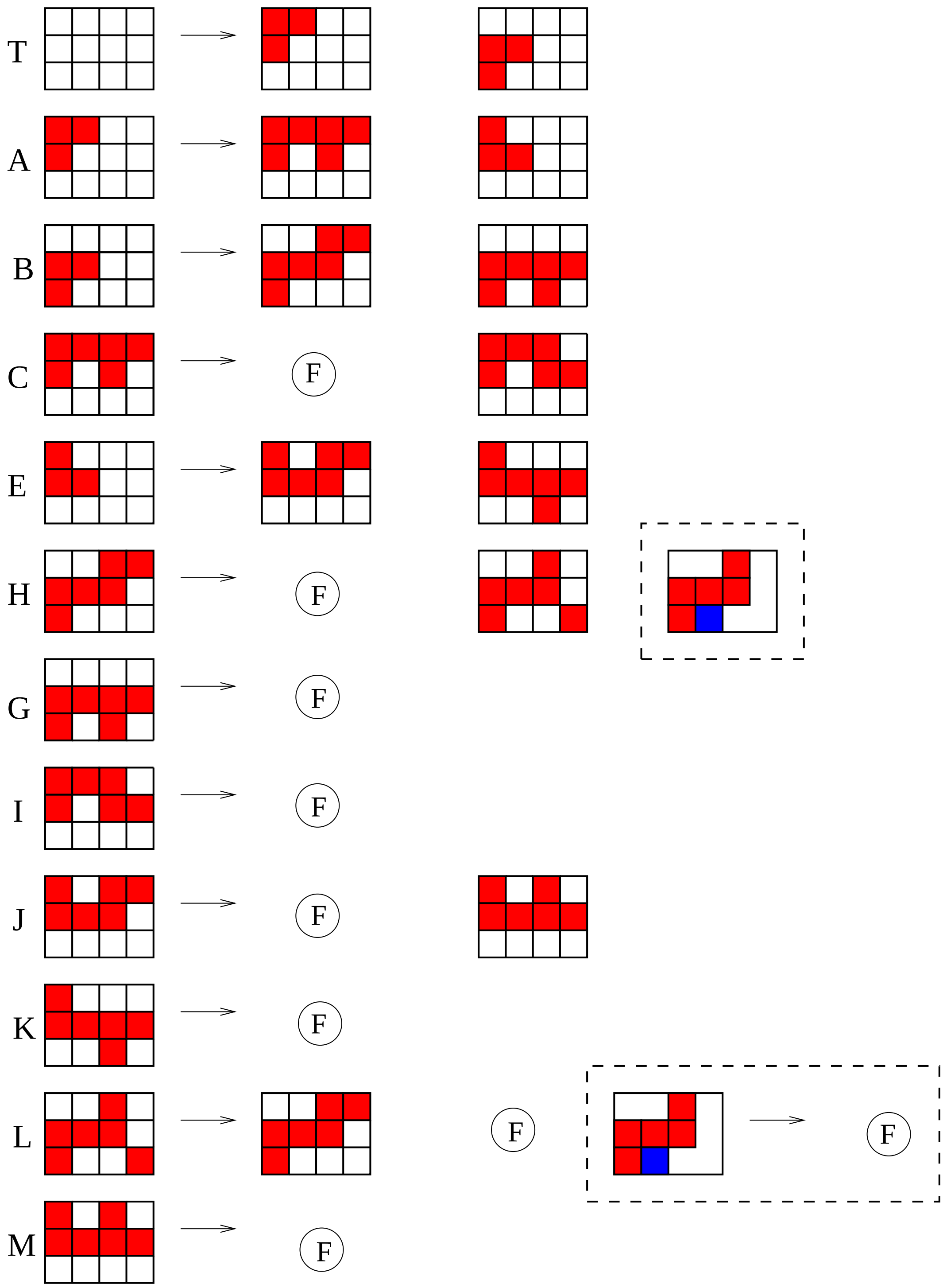
Average number of pieces:

$$M_t = 3 + M_w$$

Variance:

$$\frac{-558\,p^5 - 83\,p^4 + 116\,p^3 + 348\,p^2 + 80\,p + 16}{324 p^6}$$

Single L on the grid 3×4



Game 1

$$T = \frac{1}{2}tA + \frac{1}{2}tB, \qquad A = ptC + (1-p)\,twE$$

$$B = ptG + (1-p)\,tH, \qquad C = pt + (1-p)\,twZ$$

$$E = ptK + (1-p)\,tJ, \qquad H = pt + (1-p)\,twL$$

$$G = t, \qquad I = t, \qquad J = pt + (1-p)\,twM$$

$$K = t, \qquad L = pt + (1-p)\,twH, \qquad M = t$$

Average score:

$$M_w = \left.\frac{\partial T(t,w)}{\partial w}\right|_{t,w=1} = \frac{1 - 2\,p^2 + 2\,p^3 - p^4}{2p}$$

Average number of pieces:

$$M_t = \left.\frac{\partial T(t,w)}{\partial t}\right|_{t,w=1} = 3 + M_w$$

Variance:

$$\frac{(1-p)\left(p^7 - 3p^6 + 5p^5 + 3p^4 - 11p^3 + 13p^2 - 7p + 3\right)}{4p^2}$$

Game 2: always ends

The same system as before with $L = t$.

$$T(t,w) = \frac{1}{2}t^4wp - \frac{3}{2}p^2t^4w + \frac{1}{2}t^5w^2 - \frac{3}{2}t^5w^2p +$$

$$+\frac{3}{2}t^5w^2p^2 + \frac{1}{2}t^4wp^3 - \frac{1}{2}t^5w^2p^3 + pt^3 + \frac{1}{2}t^4w$$

Average score:

$$M_w = \left.\frac{\partial T(t,w)}{\partial w}\right|_{t,w=1} = \frac{3}{2} - \frac{5}{2}p + \frac{3}{2}p^2 - \frac{1}{2}p^3$$

Average number of pieces:

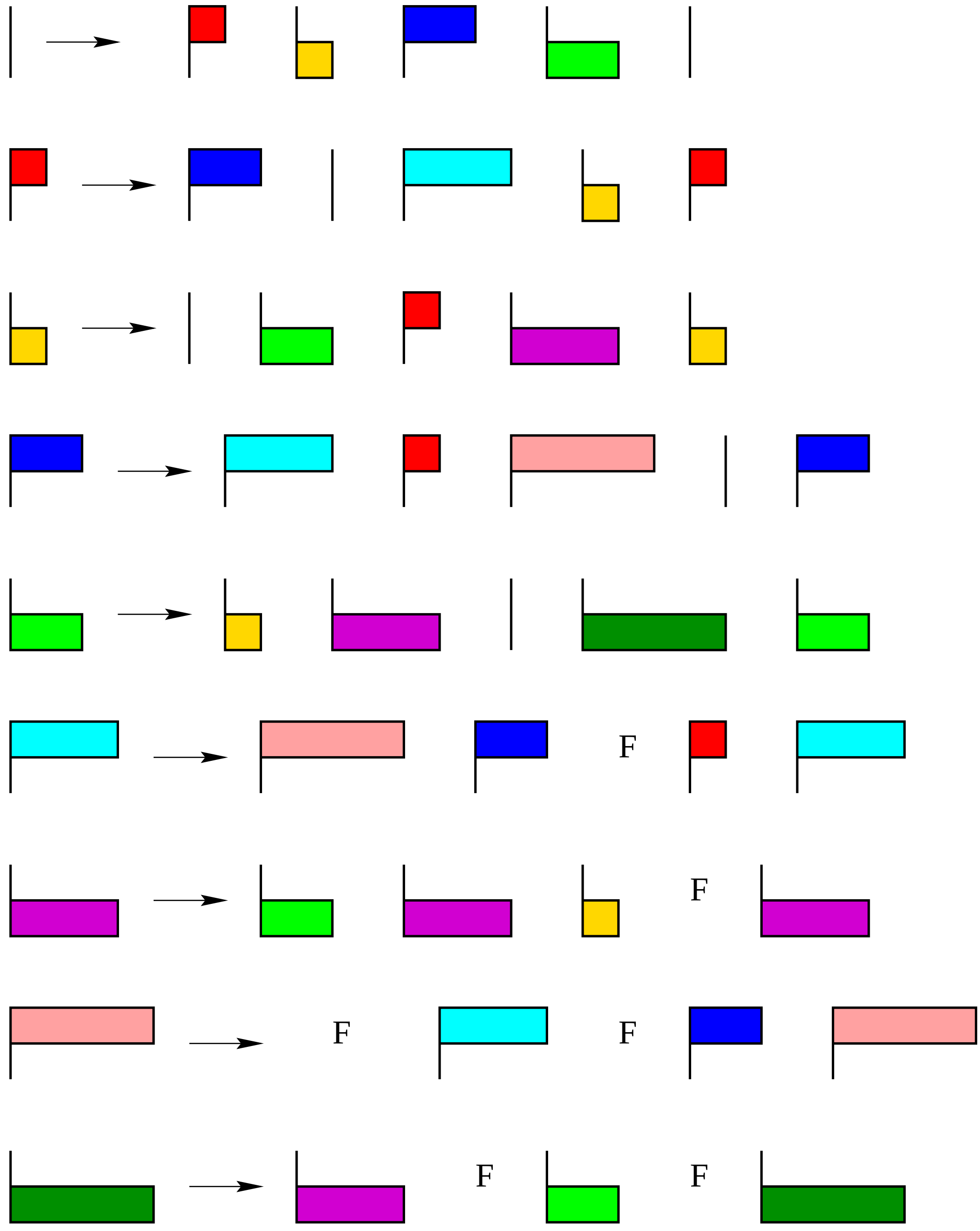
$$M_t = \left.\frac{\partial T(t,w)}{\partial t}\right|_{t,w=1} = 3 + M_w$$

Variance:

$$\frac{1}{4} + 2\,p - \frac{25}{4}\,p^2 + \frac{15}{2}p^3 - \frac{19}{4}\,p^4 + \frac{3}{2}p^5 - \frac{1}{4}p^6$$

Monomers and dimers

on the grid 2×4



Game 1

$$T = \frac{1}{5}tA + \frac{1}{5}tB + \frac{1}{5}tC + \frac{1}{5}tU + \frac{1}{5}twT$$

$$A = tpC + t\left(\frac{2}{5} - p\right)wT + ptE + t\left(\frac{2}{5} - p\right)wB + \frac{1}{5}twA$$

$$B = t\left(\frac{2}{5} - p\right)wT + tpU + t\left(\frac{2}{5} - p\right)wA + tpF + \frac{1}{5}twB$$

$$C = ptE + t\left(\frac{2}{5} - p\right)wA + tpG + t\left(\frac{2}{5} - p\right)w^2T + \frac{1}{5}twC$$

$$U = t\left(\frac{2}{5} - p\right)wB + tpF + t\left(\frac{2}{5} - p\right)w^2T + tpH + \frac{1}{5}twU$$

$$E = tpG + t\left(\frac{2}{5} - p\right)wC + tp + t\left(\frac{2}{5} - p\right)w^2A + \frac{1}{5}twE$$

$$F = t\left(\frac{2}{5} - p\right)wU + tpH + t\left(\frac{2}{5} - p\right)w^2B + tp + \frac{1}{5}twF$$

$$G = 2\,tp + t\left(\frac{2}{5} - p\right)wE + t\left(\frac{2}{5} - p\right)w^2C + \frac{1}{5}twG$$

$$H = t\left(\frac{2}{5} - p\right)wF + 2\,tp + t\left(\frac{2}{5} - p\right)w^2U + \frac{1}{5}twH$$

$$T(t,w) = \frac{8}{5}p^2t^3 - \frac{2}{25}p^2\left(15wp - 35p - 18\,w\right)t^4 + O\left(t^5\right)$$

$$M_w = \left.\frac{\partial T(t,w)}{\partial w}\right|_{t,w=1} =$$

$$-\frac{3}{25}\,\frac{625\,p^4 + 2750\,p^3 - 550\,p^2 - 256}{p^2\left(52 + 25\,p^2 + 180\,p\right)}$$

$p = 1/10 \quad 44.18861210$

$$M_t = \left.\frac{\partial T(t,w)}{\partial t}\right|_{t,w=1} = \frac{2}{5}\,\frac{500\,p^3 + 625\,p^2 + 96}{p^2\left(52 + 25\,p^2 + 180\,p\right)}$$

$p = 1/10 \quad 58.50533808$

Conclusions

- This approach allow us to study many *Tetris games*

- hopefully, Tetris . . . but the number of states is limited by $2^{(p-1)q}$ and for each oriented piece having length s we have $p-s+1$ transactions.

- Some better method?