VII International Seminar on
Analysis of Algorithms
Tatihou, France, July 2-8, 2001

Some matrices often arising in combinatorics and in the analysis of algorithms

Donatella Merlini

Dipartimento di Sistemi e Informatica
via Lombroso 6/17, 50134
University of Florence, Italy

Riordan Arrays

A Riordan Array is an infinite, lower, triangular array $\{d_{n,k}\}_{n,k\in\mathrm{N}}$, defined by two formal power series $D = (d(t), h(t))$, such that

$$d_{n,k} = [t^n]d(t)(th(t))^k.$$

It follows that $d_{n,k} = 0$ for $k > n$:

$$D = \begin{pmatrix} d_{0,0} & & & & 0 \\ d_{1,0} & d_{1,1} & & & \\ d_{2,0} & d_{2,1} & d_{2,2} & & \\ d_{3,0} & d_{3,1} & d_{3,2} & d_{3,3} & \\ \vdots & & & & \ddots \end{pmatrix}$$

For what concern the bivariate generating function, we have:

$$d(t,w) = \sum_{n,k=0}^{\infty} d_{n,k} t^n w^k =$$

$$= d(t) \sum_{k=0}^{\infty} (th(t))^k w^k = \frac{d(t)}{1 - wth(t)}.$$

We suppose $d(0) \neq 0$; if $h(0) \neq 0$ then the Riordan Array is called *proper*. Two particular situations need to be underlined:

- $d(t) = h(t)$; in this case we have a *Renewal Array* as defined in:

  D. G. Rogers. Pascal triangles, Catalan numbers and renewal arrays. Discrete Mathematics, *22:301–310, 1978.*

- $d(t) = 1$; in this case we have a correspondence with the class of convolution matrices, apart from the factor $k!/n!$.

  D. E. Knuth. Convolution polynomials. The Mathematica Journal, *4(2):67–78, 1992.*

Some more bibliography

- L. W. Shapiro, S. Getu, W.-J. Woan, and L. Woodson. The Riordan group. *Discrete Applied Mathematics*, 34:229–239, 1991.

- R. Sprugnoli. Riordan arrays and combinatorial sums. *Discrete Mathematics*, 132:267–290, 1994.

- R. Sprugnoli. Riordan arrays and the Abel-Gould identity. *Discrete Mathematics*, 142:213–233, 1995.

- D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri. On some alternative characterizations of Riordan arrays. *Canadian Journal of Mathematics*, 49(2):301–320, 1997.

The Pascal triangle

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & \mathbf{6} & \mathbf{4} & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & \mathbf{10} & 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 \\ 1 & 8 & 28 & 56 & \mathbf{70} & \mathbf{56} & 28 & 8 & 1 & 0 \\ 1 & 9 & 36 & 84 & 126 & \mathbf{126} & 84 & 36 & 9 & 1 \end{bmatrix}$$

We have:

$$d_{n,k} = \binom{n}{k} = \binom{n}{n-k} = \binom{-n+n-k-1}{n-k}(-1)^{n-k} =$$

$$= [t^{n-k}]\frac{1}{(1-t)^{k+1}} = [t^n]\frac{1}{1-t}\left(\frac{t}{1-t}\right)^k,$$

and

$$d(t) = h(t) = \frac{1}{1-t}.$$

Computing combinatorial sums

Theorem 1 *Let $D = (d(t), h(t))$ be a Riordan Array and $f(t)$ the generating function of the sequence $\{f_k\}_{k\in\mathbf{N}}$. Then:*

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n] d(t) f(th(t)).$$

For example:

$$\mathcal{G}\left\{\sum_{k=0}^{n} d_{n,k}\right\} = \frac{d(t)}{1 - th(t)}$$

$$\mathcal{G}\left\{\sum_{k=0}^{n} (-1)^k d_{n,k}\right\} = \frac{d(t)}{1 + th(t)}$$

$$\mathcal{G}\left\{\sum_{k=0}^{n} k d_{n,k}\right\} = \frac{td(t)h(t)}{(1 - th(t))^2}$$

$$\mathcal{G}\left\{\sum_{k=0}^{n} d_{n-k,k}\right\} = \frac{d(t)}{1 - t^2 h(t)}.$$

Example 1

For $a, b \in \mathbf{Z}$, $b < 0$ and $r \in \mathbf{R}$ we have:

$$\binom{r+ak}{n+bk} = [t^{n+bk}](1+t)^{r+ak} =$$

$$= [t^n]t^{-bk}(1+t)^{r+ak} = [t^n](1+t)^r \left(\frac{(1+t)^a}{t^b}\right)^k,$$

and this corresponds to the Riordan Array:

$$D = \left((1+t)^r, \frac{(1+t)^a}{t^{b+1}}\right).$$

We have:

$$\sum_k \binom{r+ak}{n+bk} f_k = [t^n](1+t)^r f(t^{-b}(1+t)^a).$$

Example 2

For $a, b, c \in \mathbf{Z}$, $b > a$ we have:

$$\binom{n+ak}{c+bk} = \binom{n+ak}{n+ak-c-bk} =$$

$$= \binom{-c-bk-1}{n+ak-c-bk}(-1)^{n+ak-c-bk} =$$

$$= [t^{n+ak-c-bk}](1-t)^{-c-bk-1}$$

$$= [t^n]t^{(b-a)k+c}(1-t)^{-c-bk-1} =$$

$$= [t^n]\frac{t^c}{(1-t)^{c+1}}\left(\frac{t^{b-a}}{(1-t)^b}\right)^k,$$

and this corresponds to the Riordan Array:

$$D = \left(\frac{t^c}{(1-t)^{c+1}}, \frac{t^{b-a-1}}{(1-t)^b}\right).$$

We have:

$$\sum_k \binom{n+ak}{c+bk} f_k = [t^n]\frac{t^c}{(1-t)^{c+1}} f\left(\frac{t^{b-a}}{(1-t)^b}\right).$$

Stirling numbers of I and II kind

The Riordan Array concept can be applied to Stirling numbers:

n	$\left[{n \atop 0}\right]$	$\left[{n \atop 1}\right]$	$\left[{n \atop 2}\right]$	$\left[{n \atop 3}\right]$	$\left[{n \atop 4}\right]$	$\left[{n \atop 5}\right]$
0	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1

n	$\left\{{n \atop 0}\right\}$	$\left\{{n \atop 1}\right\}$	$\left\{{n \atop 2}\right\}$	$\left\{{n \atop 3}\right\}$	$\left\{{n \atop 4}\right\}$	$\left\{{n \atop 5}\right\}$
0	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1

Modified Stirling numbers of the I kind

$$\mathcal{G}\left\{\frac{k!}{n!}\begin{bmatrix}n\\k\end{bmatrix}\right\} = \left(\ln\frac{1}{1-t}\right)^k$$

They correspond to the proper Riordan Array:

$$D_1 = \left(1, \frac{1}{t}\ln\frac{1}{1-t}\right).$$

Modified Stirling numbers of the II kind

$$\mathcal{G}\left\{\frac{k!}{n!}\left\{\begin{matrix}n\\k\end{matrix}\right\}\right\} = \left(e^t - 1\right)^k$$

They correspond to the proper Riordan Array

$$D_2 = \left(1, \frac{e^t - 1}{t}\right).$$

The Riordan Group

Proper Riordan Arrays constitute a group with respect to the usual row by column product between matrices. If

$$D = (d(t), h(t))$$

$$F = (f(t), g(t))$$

then the *product* $D * F$ is the proper Riordan Array:

$$D * F = (d(t)f(th(t)), h(t)g(th(t))).$$

The *identity* is the Riordan Array

$$I = (1, 1).$$

The inverse

Then we can find the *inverse* $\bar{D} = (\bar{d}(t), \bar{h}(t))$ of a proper Riordan Array with respect to the usual row by column product:

$$(d(t), h(t)) * (\bar{d}(t), \bar{h}(t)) =$$

$$= (d(t)\bar{d}(th(t)), h(t)\bar{h}(th(t))) = (1, 1),$$

that is:

$$\bar{d}(y) = \left[\frac{1}{d(t)} \middle| y = th(t) \right]$$

$$\bar{h}(y) = \left[\frac{1}{h(t)} \middle| y = th(t) \right].$$

We also have:

$$\bar{d}_{n,k} = \frac{1}{n} \left[t^{n-k} \right] \left(k - t \frac{d'(t)}{d(t)} \right) \frac{1}{d(t)h(t)^n} \qquad n > 0$$

$$\bar{d}_{0,k} = \delta_{k,0}/d_0.$$

Inverting combinatorial sums

If we have

$$\sum_{k=0}^{n} d_{n,k} f_k = g_n$$

for a given proper Riordan Array $\{d_{n,k}\}_{n,k\in\mathbf{N}}$, then we have

$$\sum_{k=0}^{n} \bar{d}_{n,k} g_k = f_n$$

for the inverse proper Riordan Array $\{\bar{d}_{n,k}\}_{n,k\in\mathbf{N}}$.

The A-sequence

Theorem 2 *A matrix* $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ *is a proper Riordan Array iff there exists a sequence* $A = \{a_i\}_{i\in\mathbf{N}}$ *with* $a_0 \neq 0$ *such that:*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots$$

Corollary 3 *Let* $D = (d(t), h(t))$ *be a proper Riordan Array and let* $A = \{a_j\}_{j\in\mathbf{N}}$ *be the corresponding* A*-sequence. Then, if* $A(t)$ *is the generating function of the sequence* A*, we have:*

$$h(t) = A(th(t)).$$

The Catalan triangle

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{3} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 14 & \mathbf{14} & 9 & 4 & 1 & 0 & 0 & 0 & 0 \\ 42 & 42 & 28 & 14 & 5 & 1 & 0 & 0 & 0 \\ 132 & 132 & \mathbf{90} & \mathbf{48} & \mathbf{20} & \mathbf{6} & \mathbf{1} & 0 & 0 \\ 429 & 429 & 297 & \mathbf{165} & 75 & 27 & 7 & 1 & 0 \\ 1430 & 1430 & 1001 & 572 & 275 & 110 & 35 & 8 & 1 \end{bmatrix}$$

$$d(t) = h(t) = \frac{1 - \sqrt{1 - 4t}}{2t}$$

$$A(t) = \frac{1}{1 - t}$$

The Motzkin triangle

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 7 & 3 & 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 15 & \mathbf{14} & 12 & \mathbf{4} & \mathbf{5} & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 \\ 36 & 37 & \mathbf{24} & 18 & 5 & 6 & 0 & 1 & 0 & 0 \\ 91 & 90 & 67 & 36 & 25 & 6 & 7 & 0 & 1 & 0 \\ 232 & 233 & 165 & 106 & 50 & 33 & 7 & 8 & 0 & 1 \end{bmatrix}$$

$$d(t) = h(t) = \frac{1 + t - \sqrt{(1-3t)(1+t)}}{t(1+t)}$$

$$A(t) = \frac{1 - t + t^2}{1 - t}$$

The modified Stirling triangle (first kind)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & \frac{11}{12} & 3/2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1/5 & 5/6 & 7/4 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1/6 & \frac{137}{180} & \frac{15}{8} & \frac{17}{6} & 5/2 & 1 & 0 & 0 \\ 0 & 1/7 & \frac{7}{10} & \frac{29}{15} & 7/2 & \frac{25}{6} & 3 & 1 & 0 \\ 0 & 1/8 & \frac{363}{560} & \frac{469}{240} & \frac{967}{240} & \frac{35}{6} & \frac{23}{4} & 7/2 & 1 \end{bmatrix}$$

$$d_{n,k} = \frac{k!}{n!} \begin{bmatrix} n \\ k \end{bmatrix} \qquad D_1 = \left(1, \frac{1}{t} \ln \frac{1}{1-t}\right),$$

$$A(y) = \frac{ye^y}{e^y - 1} = 1 + \frac{1}{2}y + \frac{1}{12}y^2 - \frac{1}{720}y^4 + \cdots$$

The modified Stirling triangle (second kind)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/24 & \frac{7}{12} & 3/2 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{120} & 1/4 & 5/4 & 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{720} & \frac{31}{360} & 3/4 & \frac{13}{6} & 5/2 & 1 & 0 & 0 \\ 0 & \frac{1}{5040} & 1/40 & \frac{43}{120} & 5/3 & 10/3 & 3 & 1 & 0 \\ 0 & \frac{1}{40320} & \frac{127}{20160} & \frac{23}{160} & \frac{81}{80} & \frac{25}{8} & \frac{19}{4} & 7/2 & 1 \end{bmatrix}$$

$$d_{n,k} = \frac{k!}{n!} \left\{ {n \atop k} \right\} \qquad D_2 = \left(1, \frac{e^t - 1}{t}\right)$$

$$A(y) = \frac{y}{\ln(1+y)} = 1 + \frac{1}{2}y - \frac{1}{12}y^2 + \frac{1}{24}y^3 - \frac{19}{720}y^4 + \cdots$$

The Z-sequence

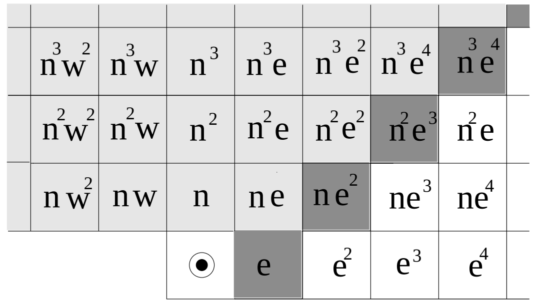
Theorem 4 *If $\{d_{n,k}\}_{n,k\in\mathbf{N}}$ is a proper Riordan Array then a unique sequence $Z = \{z_0, z_1, z_2, \ldots\}$ there exists such that*

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots$$

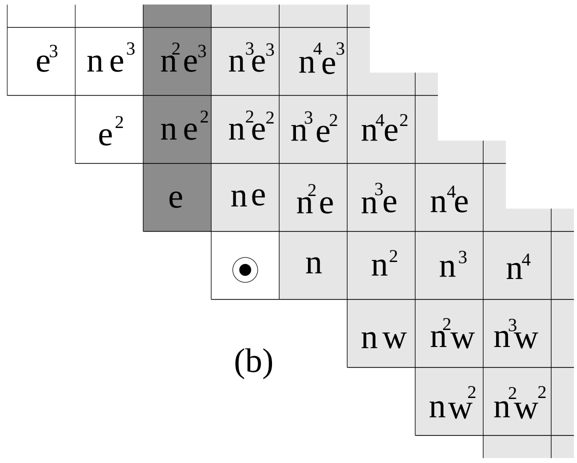
Theorem 5 *Let $(d(t), h(t))$ be a proper Riordan Array and let $Z(t)$ be the generating function of the Z-sequence. Then we have:*

$$d(t) = \frac{d_0}{1 - tZ(th(t))}.$$

Lattice paths and Riordan Arrays



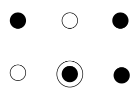
(a)



(b)

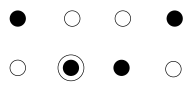
The arrays corresponding to paths with steps $(0,1)$, $(1,0)$ and i) $(1,2)$, ii) $(1,3)$, iii) $(2,1)$, iv) $(3,1)$ respectively.

n/k	0	1	2	3	4
0	1				
1	1	1			
2	3	2	1		
3	9	7	3	1	
4	31	24	12	4	1



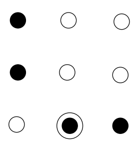
(i)

n/k	0	1	2	3	4
0	1				
1	1	1			
2	2	2	1		
3	6	5	3	1	
4	19	16	9	4	1



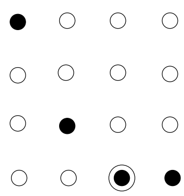
(ii)

n/k	0	1	2	3	4
0	1				
1	1	1			
2	3	3	1		
3	9	9	5	1	
4	31	31	19	7	1



(iii)

n/k	0	1	2	3	4
0	1				
1	1	1			
2	2	2	1		
3	6	6	4	1	
4	19	19	13	6	1



(iv)

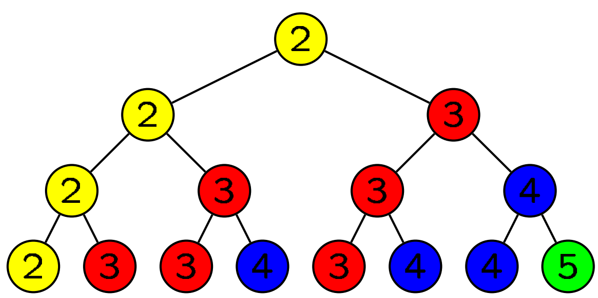
GENERATING TREES

In D. Merlini and M. C. Verri. Generating trees and proper Riordan Arrays. *Discrete Mathematics*, 218: 167–183, 2000 it is shown that a particular subset of generating trees has a correspondence with some proper Riordan array.

DEFINITION An infinite matrix $\{d_{n,k}\}_{n,k\in\mathbb{N}}$ is *associated to a generating tree* with root (c) (AGT matrix for short) if $d_{n,k}$ is the number of nodes at level n with label $k + c$.

Under suitable conditions, this matrix corresponds to a proper Riordan Array, and vice versa.

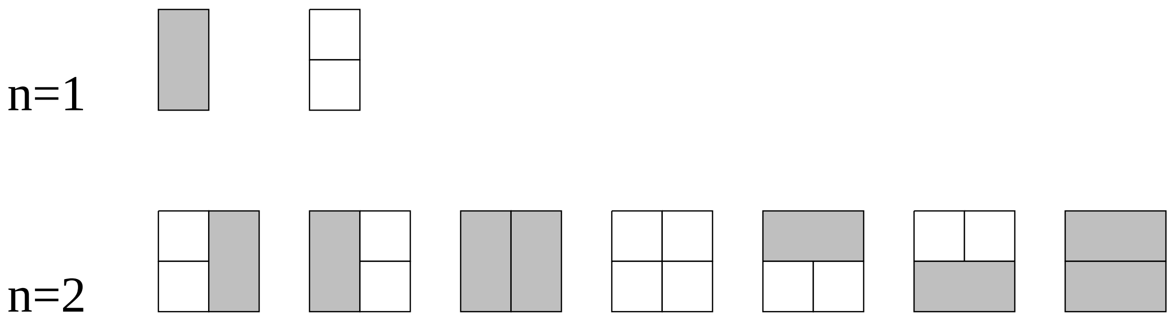
$$\begin{cases} root: & (2) \\ rule: & (k) \quad \to (k)(k+1) \end{cases}$$



This corresponds to the Pascal triangle.

Monomer-dimer tiling

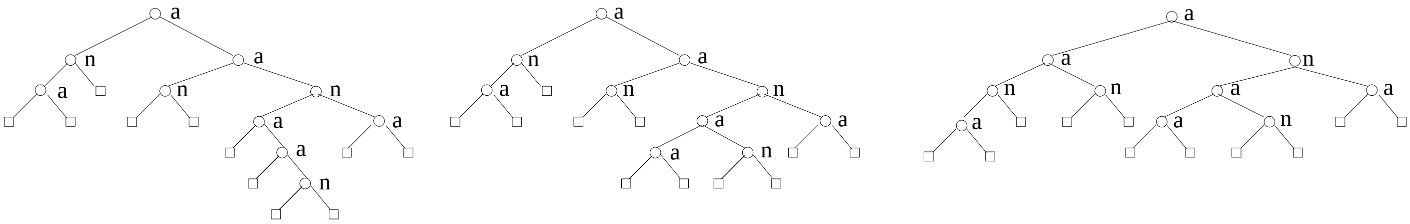
Let $d_{n,k}$ denotes the number of tilings of a $2 \times n$ strip by means of monomers and dimers containing exactly $2k$ monomers.



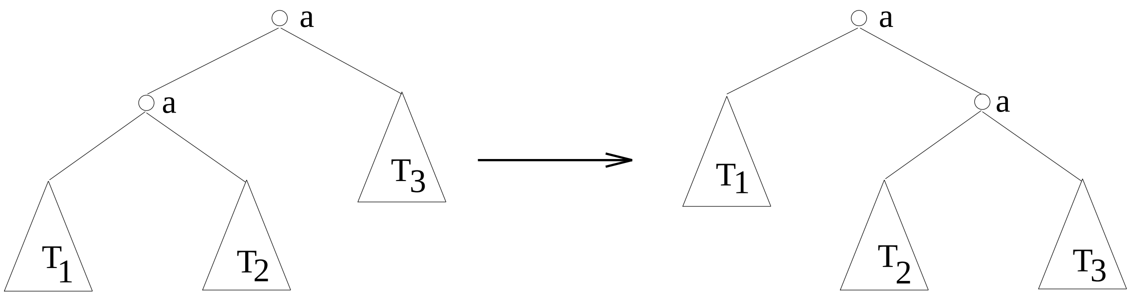
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 11 & 7 & 1 & 0 & 0 & 0 & 0 \\ 5 & 26 & 29 & 10 & 1 & 0 & 0 & 0 \\ 8 & 56 & 94 & 56 & 13 & 1 & 0 & 0 \\ 13 & 114 & 263 & 234 & 92 & 16 & 1 & 0 \\ 21 & 223 & 667 & 815 & 473 & 137 & 19 & 1 \end{bmatrix}$$

$$d(t) = \frac{1}{1-t-t^2} \qquad h(t) = \frac{(1+t)}{(1-t)(1-t-t^2)}$$

Hybrid trees



Three $\{\mathbf{a}, \mathbf{n}\}$ binary trees which are equivalent for the associative property. The first is the representative of the equivalence class.



An associative node

Let $S_{n,k}$ be the number of hybrid trees with n internal nodes, with k letters **a** or **n** ($1 \leq k \leq n$) before the leftmost **e** in the corresponding codeword.

Let c_m be the number of words of length m : $T_{n,k} = S_{n,k}/c_k$ is a Riordan array.

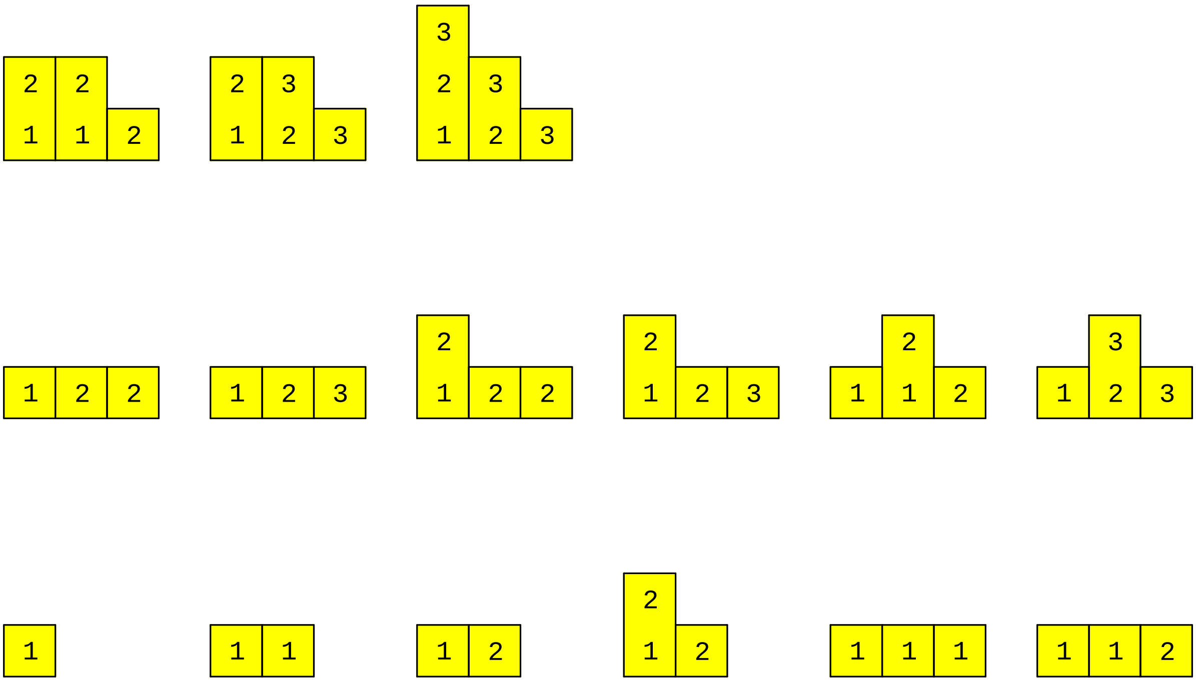
The triangle for hybrid trees

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 31 & 18 & 6 & 1 & 0 & 0 & 0 & 0 \\ 154 & 90 & 33 & 8 & 1 & 0 & 0 & 0 \\ 820 & 481 & 185 & 52 & 10 & 1 & 0 & 0 \\ 4575 & 2690 & 1065 & 324 & 75 & 12 & 1 & 0 \\ 26398 & 15547 & 6276 & 2006 & 515 & 102 & 14 & 1 \end{bmatrix}$$

$$A(t) = \frac{1+t}{1-t-t^2}$$

$$Z(t) = \frac{2+t}{1-t-t^2}$$

A model for a printer



Let $\mathcal{C}_{n,k\in\mathbf{N}}$ be the number of schedules of length n with k jobs' requests at the first time slot; then we have:

$$\mathcal{C}_{n+1,k+1} = \mathcal{C}_{n,k} + 2\mathcal{C}_{n,k+1} + 2\mathcal{C}_{n+1,k+2} + \ldots$$

and

$$\mathcal{C}_{n+1,1} = 2\mathcal{C}_{n,1} + 2\mathcal{C}_{n+1,2} + \ldots,$$

that is, $\{\mathcal{C}_{n,k}\}_{n,k}$ represents a proper Riordan Array having A-sequence $A = \{1, 2, 2, \ldots\}$ and Z-sequence $Z = \{2, 2, \ldots\}$.

The triangle for the model

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{22} & \mathbf{16} & \mathbf{6} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 90 & \mathbf{68} & 30 & 8 & 1 & 0 & 0 & 0 \\ 394 & 304 & 146 & 48 & 10 & 1 & 0 & 0 \\ 1806 & 1412 & 714 & 264 & 70 & 12 & 1 & 0 \\ 8558 & 6752 & 3534 & 1408 & 430 & 96 & 14 & 1 \end{bmatrix}$$

$$d(t) = h(t) = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2t}$$

$$A(t) = \frac{1 + t}{1 - t}$$