The area determined by underdiagonal lattice paths

Donatella Merlini, Renzo Sprugnoli, M. Cecilia Verri

Dipartimento di Sistemi e Informatica,
via Lombroso 6/17, Firenze, Italy,
resp@ingfi1.ing.unifi.it

Abstract. We use the "first passage decomposition" methodology to study the area between various kinds of underdiagonal lattice paths and the main diagonal. This area is important because it is connected to the number of inversions in permutations and to the internal path length in various types of trees. We obtain the generating functions for the total area of all the lattice paths from the origin to the point (n, n). Since this method also determines the number of these paths, we are able to obtain exact results for the average area.

Keywords: underdiagonal lattice paths, average area, first passage decomposition, generating functions, context-free grammars.

1 Introduction

In the present paper, we investigate the area determined by various types of underdiagonal lattice paths and the main diagonal $x = y$. We examine a lattice path scheme, i.e., a set $R = \{(\delta_i, \delta_i', \kappa_i) \mid i = 1, 2, \cdots, r\}$, for some positive integer r, of coloured step templates (see next section), and the set D_n of the R-paths from $(0,0)$ to (n, n), i.e., the underdiagonal paths conforming to the scheme R; we denote the total area between all the $|D_n|$ R-paths and the main diagonal by A_n. For example, we give a picture of the shortest paths corresponding to the scheme $R = \{(1, 0, \mathit{black}), (0, 1, \mathit{black})\}$ in Figure 1(a) and we show how the area is computed in Figure 1(b).

By performing some easy manual computations, we obtain the following first values: $A_0 = 0, A_1 = 1/2, A_2 = 3, A_3 = 29/2$, and so on, (see Figure 2).

We will continue to use the notations in Figure 1(a). Another way of looking at lattice paths consists in rotating the path around the main diagonal and then rotating the new figure around the origin by 45°. We thus obtain the picture in Figure 1(c), illustrating the so-called "French notation" for lattice paths. This kind of notation may seem more appealing from an intuitive point of view but it is substantially the same as ours.

We can consider this problem in lattice path theory as being autonomous (see, e.g., [27]), but it actually has some important connexions to other problems in combinatorics and computer science. For example, the area is related

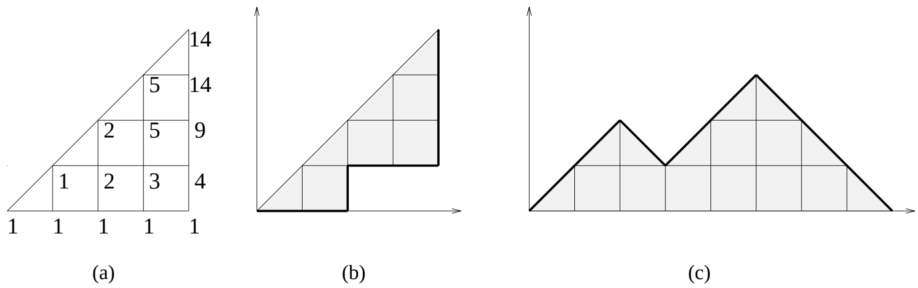

Fig. 1. Underdiagonal lattice paths conforming to the scheme $R = \{(1, 0, \mathit{black}),$ $(0, 1, \mathit{black})\}$.

to permutations (see, e.g., Knuth [17], vol. III, p.87); besides, the recurrences obtained by studying the area are the same as those found in the analysis of the internal path length of various kinds of trees. The problem of lattice path area has been studied in a q-analogue context because q-formulas arise when we consider the area together with some other parameters, such as the path length; we mention here [2, 3, 4, 11]. Finally, we wish to quote the paper by Francon and Viennot [8] on which part of the exposition in [15] is based.

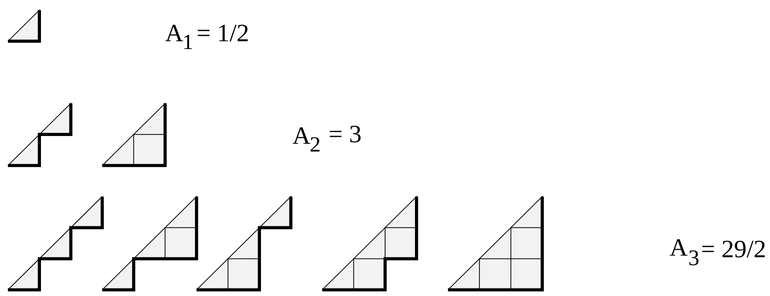

Fig. 2. The values A_1, A_2, A_3 for the scheme $R = \{(1, 0, \mathit{black}), (0, 1, \mathit{black})\}$.

2 First Passage Decomposition

Let $\Pi = \mathbf{Z}^2$ be the integral lattice. A *step template* is a triple $(\delta, \delta', \kappa)$ where $\delta, \delta' \in \mathbf{Z}$ and κ belongs to a (possible infinite) set of *colours*. A *coloured* $(\delta, \delta', \kappa)$-

step is a triple $((x,y),(x+\delta,y+\delta'),\kappa)$ where (x,y) and $(x+\delta,y+\delta')$ are two points in Π. A *path scheme* W is a (finite) set W_s of step templates together with a set W_c of *conditions* on the steps (e.g., certain kinds of steps cannot immediately follow other kinds of steps; some steps are allowed or not allowed when they reach the main diagonal; the whole path must stay on or below the main diagonal; and so on). A *W-path* is a finite sequence of coloured steps $(s_1,s_2,\ldots,s_m)$ such that:

a) $s_1=((0,0),(\delta,\delta'),\kappa)$ with $(\delta,\delta',\kappa)\in W_s$;
b) $\forall i=2,3,\ldots,m-1$, if $s_i=((x_i,y_i),(x_i+\delta,y_i+\delta'),\kappa)$ and $s_{i+1}=((x_{i+1},y_{i+1}),(x_{i+1}+\overline{\delta},y_{i+1}+\overline{\delta}'),\overline{\kappa})$, then $(\delta,\delta',\kappa)\in W_s$, $(\overline{\delta},\overline{\delta}',\overline{\kappa})\in W_s$ and $x_i+\delta=x_{i+1}, y_i+\delta'=y_{i+1}$;
c) all the steps $(s_1,s_2,\ldots,s_m)$ satisfy the conditions in W_c.

The number m is the *length* of the W-path. When the set of colours consists in only one element (i.e., all the steps are the same colour, say *black*), a path can be described more simply by giving the ordered $(m+1)$-tuple of points $(O,P_1,P_2,\ldots,P_m)$ the path passes through. Because of its simplicity, this definition is very common; however, when dealing with coloured paths, a step (or step template) approach seems to be more appropriate. Finally, a *path model* is a set of path schemes having a common set W_c of conditions.

A step with a template $(0,1,\kappa)$ is called a *north-step* and east, south and west steps are defined in an analogous way. A north-east, or *diagonal*, step is any step with a step template $(1,1,\kappa)$.

Many kinds of path models have been studied in literature, (see [9, 10, 13, 14, 15, 16, 18, 19, 21, 22, 23, 24, 25]). In this paper, we are interested in a model called the $\mathcal{R}$-model. The paths are made up of step templates (δ,δ',κ) with $\delta,\delta'\geq 0$, and never go above the main diagonal. More formally, a scheme R in the $\mathcal{R}$-model can be identified with the set of its templates:

$$R=\{(\delta_i,\delta_i',\kappa_i)\mid \delta_i,\delta_i'\geq 0, i=1,2,\cdots,r\},\quad \text{for some positive integer } r,$$

the only condition for the paths being to remain on or below the main diagonal.

We use the case in Figure 1 as an introductory example of our approach to lattice path area problems. Let D be the set of all possible R-paths, i.e. those starting from the origin $O\equiv(0,0)$, arriving at a point (n,n) and made up of only steps with a template $(1,0,\textit{black})$ or $(0,1,\textit{black})$. If $\gamma=OA$ is any R-path, let P be the first point at which γ touches the diagonal $x=y$ again. It can occur that $P\equiv A$ and, if the path is empty, $P\equiv A\equiv O$. In this way (see Figure 3), when γ is not empty, it can be decomposed into the following parts:

- the first step OB, whose template is always $(1,0,\textit{black})$;
- a subpath BQ, which is a path in D, according to the definition of P, translated one position to the right (BQ can touch the diagonal $x-y=1$, but can never go above it);
- a step QP, whose template is always $(0,1,\textit{black})$;
- a subpath PA, which is also a path in D.

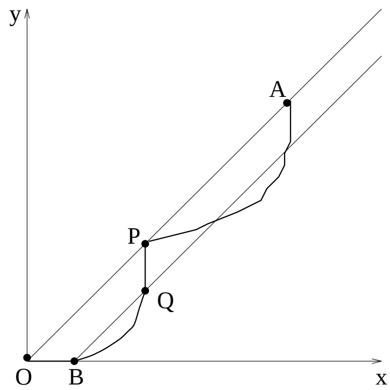

Fig. 3. The decomposition of a path in the scheme $R = \{(1,0,\mathit{black}),(0,1,\mathit{black})\}$.

By borrowing some notations from the theory of context-free languages or "object grammars" (see [6, 12]), we can write the following definition:

$$D ::= \epsilon \mid aDbD \tag{1}$$

where $a = (1,0,\mathit{black})$ and $b = (0,1,\mathit{black})$. In a paper of ours [20], we showed how this approach, called "first passage decomposition" methodology (see, e.g., Feller [7]), can give us some very general results for lattice paths counting problems. Here, we wish to show that the same approach can often be used to derive results in lattice path area problems.

Theorem 2.1 *The total area determined by the* $\frac{1}{n+1}\binom{2n}{n}$ *Dyck paths and the main diagonal is asymptotics to* $4^n/2$ *and therefore the corresponding average value is* $0.5\sqrt{\pi}n^{3/2}$.

Proof:
Formula (1) is the definition of the Dyck language. Let us set $d_n = |D_n|$, where D_n is the set of lattice paths which start at the origin and arrive at point (n,n). Therefore, we have $D = \bigcup_{n\in N} D_n$ and, by Schützenberger's methodology (e.g., [5, 26]), we can derive the following expression from (1):

$$d_{n+1} = \sum_{k=0}^{n} d_k d_{n-k},$$

or the functional equation:

$$d(t) = 1 + td(t)^2,$$

where $d(t)$ is the generating function $d(t) = \sum_{k=0}^{\infty} d_k t^k$. It is now easy for us to obtain an explicit expression for the generating function and a formula for d_n:

$$d(t) = \frac{1-\sqrt{1-4t}}{2t}, \qquad d_n = \frac{1}{n+1}\binom{2n}{n}. \tag{2}$$

We recognize the n^{th} *Catalan number* in d_n. If we now denote as A_n the total area of all the d_n paths in D_n, from (1) we can deduce a recursive relation for A_n. Let us use Figure 3 as a reference; when we have $A \equiv (n+1, n+1)$ and $P \equiv (k+1, k+1)$, the total area between the path γ and the main diagonal is the sum of three parts:

i) the area of the trapezium $OBQP$, which is $k + 1/2$;
ii) the area between the subpath BQ and the diagonal $x - y = 1$; we denote the area by a_k, which is one of the components of A_k;
iii) the area between the subpath PA and the diagonal $x - y = 0$; we denote the area by a_{n-k}, which is one of the components of A_{n-k};

If we now examine all the paths in D_{n+1}, we have to sum up the contributions of the three above areas:

- the area in i) is contained in every path $\gamma \in D_{n+1}$. If we sum with respect to k, we find a total contribution of $\sum_{k=0}^{n}(k+1/2)d_k d_{n-k}$;
- the area a_k in ii) appears as often as the possible subpaths in D_{n-k} between P and A. This happens for all the possible subpaths in D_k, which have a total area A_k. Therefore the total contribution of these paths is $\sum_{k=0}^{n} A_k d_{n-k}$;
- in a similar way, the total contribution of the subpaths PA is $\sum_{k=0}^{n} A_{n-k} d_k$.

By summing up all these contributions, we obtain:

$$A_{n+1} = \sum_{k=0}^{n}(k+\frac{1}{2})d_k d_{n-k} + \sum_{k=0}^{n} A_k d_{n-k} + \sum_{k=0}^{n} A_{n-k} d_k.$$

The last two sums are equal, and so we have:

$$A_{n+1} = 2\sum_{k=0}^{n} A_k d_{n-k} + \frac{1}{2}\sum_{k=0}^{n} d_k d_{n-k} + \sum_{k=0}^{n} k d_k d_{n-k}.$$

Since the initial condition is $A_0 = 0$, we now go on to the generating functions. In fact, $d(t)$ is given by formula (2) and the generating function for kd_k is obviously $td'(t)$:

$$\frac{A(t)}{t} = 2A(t)d(t) + \frac{1}{2}d(t)^2 + td'(t)d(t).$$

By differentiating $d(t)$ and solving in $A(t)$ we get:

$$A(t) = \frac{1 - 2t - \sqrt{1-4t}}{4t(1-4t)}.$$

We can now obtain the value for A_n:

$$A_n = [t^n]A(t) = \frac{1}{4}[t^{n+1}]\frac{1}{1-4t} - \frac{1}{2}[t^n]\frac{1}{1-4t} - \frac{1}{4}[t^{n+1}]\frac{1}{\sqrt{1-4t}} =$$

$$= \frac{4^n}{2} - \frac{1}{4}\binom{2n+2}{n+1} \sim \frac{4^n}{2}\left(1 - \frac{2}{\sqrt{\pi(n+1)}} + O\left(\frac{1}{n}\right)\right).$$

Finally, by dividing this expression by $d_n = \frac{1}{n+1}\binom{2n}{n}$, (the total number of lattice paths), we find that the average value for the area is asymptotic to $0.5\sqrt{\pi}n^{3/2}$. ■

This first passage decomposition method is very general and can be used for solving many counting problems. It can easily be applied to lattice path area problems when the scheme $R = \{(\delta_i, \delta_i', \kappa_i) \mid i = 1, 2, \cdots, r\}$ only contains step templates having $|\delta_i - \delta_i'| \leq 1$ (see next section). In this case, there can always be a decomposition (see Figure 3), and the area of trapezium $OBQP$ can easily be computed. We can show this by another example.

Example 2.2 Let us examine a scheme with step templates $R = \{(1, 0, \mathit{black}), (0, 1, \mathit{black}), (1, 2, \mathit{black})\}$. Figure 4 illustrates the first passage decompositions

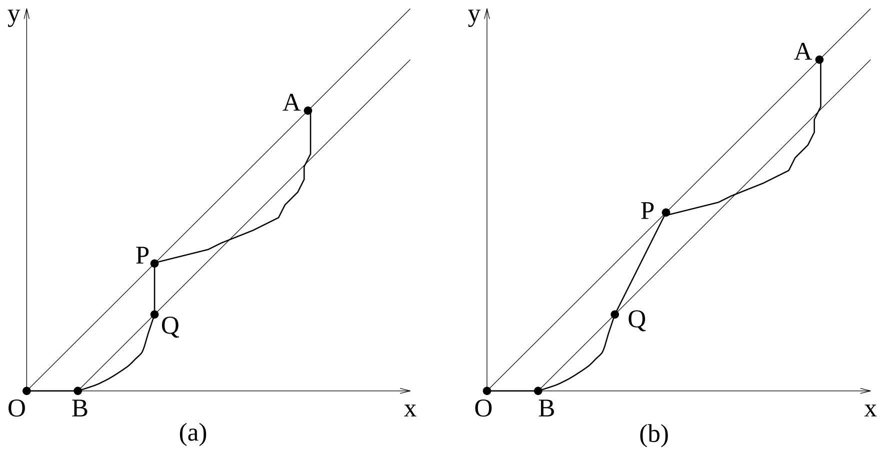

Fig. 4. First passage decompositions for a path in the scheme $R = \{(1, 0, \mathit{black}), (0, 1, \mathit{black}), (1, 2, \mathit{black})\}$.

we can have. It also shows that they contribute to the area determined by a path from the origin to a point $(n+2, n+2)$ in the following way :

- Fig. 4(a): $A_k d_{n+1-k} + A_{n+1-k} d_k + \left(k + \frac{1}{2}\right) d_k d_{n+1-k}$;
- Fig. 4(b): $A_k d_{n-k} + A_{n-k} d_k + (k+1)\, d_k d_{n-k}$.

Therefore, we have the following recurrence relation:

$$A_{n+2} = 2\sum_{k=0}^{n+1} A_k d_{n+1-k} + \sum_{k=0}^{n+1} \left(k + \frac{1}{2}\right) d_k d_{n+1-k} + \\ + 2\sum_{k=0}^{n} A_k d_{n-k} + \sum_{k=0}^{n} (k+1)\, d_k d_{n-k}. \qquad (3)$$

The function $d(t)$ is easily determined by the same decomposition, which gives the defining formula:

$$D ::= \epsilon \mid aDbD \mid aDcD,$$

where D is the set of lattice paths under consideration and a, b, c are the three step templates $(1, 0, black), (0, 1, black)$ and $(1, 2, black)$, respectively. By applying Schützenberger's methodology, we find:

$$d(t) = 1 + td(t)^2 + t^2 d(t)^2,$$

and then easily obtain:

$$d(t) = \frac{1 - \sqrt{1 - 4t - 4t^2}}{2t(1+t)}.$$

The asymptotic value of d_n can be simply computed and we have:

$$d_n \sim \sqrt{\frac{2\sqrt{2}}{1+\sqrt{2}}} \frac{(2+\sqrt{2})^n}{n\sqrt{\pi n}}.$$

As far as the area problem is concerned, we note that the initial conditions are $A_0 = 0, A_1 = 1/2$. We can rewrite relation (3) in terms of generating functions:

$$\frac{A(t) - t/2}{t^2} = 2\frac{A(t)d(t)}{t} + \frac{d(t)^2 - 1}{2t} + d(t)d'(t) +$$
$$+2A(t)d(t) + d(t)^2 + td(t)d'(t).$$

By differentiating $d(t)$ and performing the necessary computations, we eventually find:

$$A(t) = \frac{1}{4} \frac{(1+2t)\left(1 - 2t - 2t^2 - \sqrt{1 - 4t - 4t^2}\right)}{t(1 - 4t - 4t^2)(t+1)^2},$$

and the asymptotic value for A_n is $(2+\sqrt{2})^n/2$.

3 The area of a particular class of paths

We now want to generalize the results described in the previous section and show that the method of first passage decomposition can be used to find the area A_n of all paths from $(0, 0)$ to (n, n), according to the scheme:

$$R' = \{(\delta_i, \delta'_i, \kappa_i) \mid \mid \delta_i - \delta'_i \mid \leq 1, i = 1, 2, \ldots, r'\},$$

for some $r' > 0$.

If we define:

$$S = \{(\delta_i, \delta'_i, \kappa_i) \in R' \mid \delta_i = \delta'_i + 1, i = 1, 2, \ldots, r\},$$
$$\overline{S} = \{(\overline{\delta}_j, \overline{\delta}'_j, \overline{\kappa}_j) \in R' \mid \overline{\delta}_j = \overline{\delta}'_j - 1, j = 1, 2, \ldots, \overline{r}\},$$
$$\widehat{S} = \{(\widehat{\delta}_h, \widehat{\delta}'_h, \widehat{\kappa}_h) \in R' \mid \widehat{\delta}_h = \widehat{\delta}'_h, h = 1, 2, \ldots, \widehat{r}\},$$

with $r + \overline{r} + \widehat{r} = r'$, then R' is the disjoint union of these sets:

$$R' = S \cup \overline{S} \cup \widehat{S}.$$

Theorem 3.1 *Let R' be a scheme as above. There exists a second degree equation defining the generating function $d(t)$ counting the number d_n of underdiagonal paths starting at the origin and arriving to the point (n,n) on the main diagonal. Furthermore, a recurrence relation exists espressing the area A_n determined by these paths and the main diagonal.*

Proof:
Let D be the set of all possible R'-paths. If $\gamma = OA \in D$ is any R'-path, let P be the first point at which γ touches the diagonal $x = y$. We can have $P \equiv A$ and, if the path is empty, $P \equiv A \equiv O$.

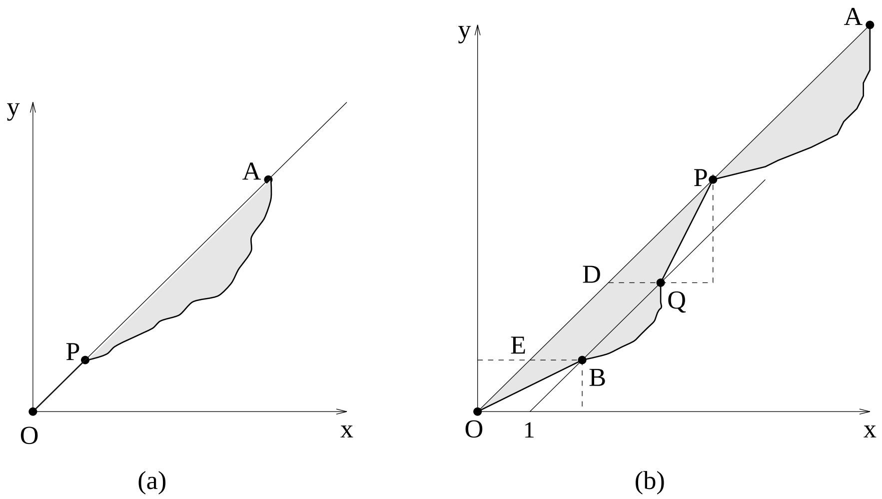

Fig. 5. The first passage decompositions for a path in the R' scheme.

When γ is not empty, two possible situations can occur:

a) the first step OB belongs to $\widehat{S}$; in this case, $B \equiv P$ and γ can be decomposed into the following parts:
   - the first step OP;
   - a subpath PA, which is also a path in D.

   This applies to all the steps in $\widehat{S}$;

b) the first step OB belongs to S; in this case, γ can be decomposed into the following parts:
   - the first step OB belonging to S;
   - a subpath BQ, which is a path in D translated one position to the right;
   - a step QP belonging to $\overline{S}$;
   - a subpath PA which is a path in D.

   This applies to all the steps $OB \in S$ and $QP \in \overline{S}$.

As a result, we have the following definition:

$$D ::= \epsilon \mid \widehat{s}_1 D \mid \cdots \mid \widehat{s}_{\hat{r}} D \mid s_1 D \overline{s}_1 D \mid \cdots \mid s_i D \overline{s}_j D \mid \cdots \mid s_r D \overline{s}_{\overline{r}} D. \qquad (4)$$

We can apply Schützenberger's methodology, and derive a functional equation for the function $d(t) = \sum d_n t^n$ from (4), where $d_n = |D_n|$ and $D = \bigcup_{n \in N} D_n$ (as previously). Every term $\widehat{s}_h D$ gives a contribution equal to $t^{\widehat{\delta}_h} d(t)$ and every term $s_i D \overline{s}_j D$ contributes for $t^{\delta_i + \overline{\delta}_j} d(t)^2$, i.e.

$$d(t) = 1 + \sum_{h=1}^{\hat{r}} t^{\hat{\delta}_h} d(t) + \sum_{i=1}^{r} \sum_{j=1}^{\overline{r}} t^{\delta_i + \overline{\delta}_j} d(t)^2. \qquad (5)$$

We wish to point out that (5) is a second degree equation that can always be solved for $d(t)$.

We can now deduce a recurrence relation for A_n, the total area of all the d_n paths in D_n. Let us examine a path $\gamma = OA$ in D and use Figure 5 as a reference. We assume that $A \equiv (n+v, n+v)$, where $v = max_{i,j}\{\delta_i + \overline{\delta}_j\}$. The total area A_{n+v} between the path γ and the main diagonal is the sum of the following elements:

a) if γ is of type (a) (Fig. 5), the area we are looking for is the area of the path PA, i.e. a path in $D_{n+v-\hat{\delta}_h}$, for all step templates in $\widehat{S}$;
b) if γ is of type (b) (Fig. 5) and we assume that the distance from B to Q, measured along the x or y axis, is k, the area is the sum of the following three parts:
   b1) the area of the trapezium $OBQP$, which is $k + (\delta'_i + \overline{\delta}'_j)/2$, because triangles OBE and DQP have areas equal to $\delta'_i/2$ and $\overline{\delta}'_j/2$, respectively;
   b2) the area between subpath BQ and the diagonal $x - y = 1$;
   b3) the area between subpath PA and the main diagonal.

The total area A_{n+v} is obtained by summing up all the contributions of the previous areas, for all the paths in D_{n+v} :

- the area corresponding to point a) is $A_{n+v-\hat{\delta}_h}$;
- the area corresponding to point b1) must be counted for all possible combinations $d_k d_{n+v-(\delta_i+\overline{\delta}_j)-k}$ of paths BQ and PA as k varies from 0 to $n+v-(\delta_i+\overline{\delta}_j)$, and we obtain:

$$\sum_{k=0}^{n+v-(\delta_i+\overline{\delta}_j)} \left(k + \frac{\delta'_i + \overline{\delta}'_j}{2} \right) d_k d_{n+v-(\delta_i+\overline{\delta}_j)-k};$$

- the area corresponding to point b2) must be counted for all possible paths PA in $D_{n+v-(\delta_i+\overline{\delta}_j)-k}$ as k varies from 0 to $n+v-(\delta_i+\overline{\delta}_j)$, and we obtain:

$$\sum_{k=0}^{n+v-(\delta_i+\overline{\delta}_j)} A_k d_{n+v-(\delta_i+\overline{\delta}_j)-k};$$

- the area corresponding to point b3) must be counted for all possible paths PA in D_k as k varies from 0 to $n+v-(\delta_i+\overline{\delta}_j)$, and we obtain:

$$\sum_{k=0}^{n+v-(\delta_i+\overline{\delta}_j)} A_{n+v-(\delta_i+\overline{\delta}_j)-k} d_k.$$

By summing up all these contributions, we obtain the following recurrence relation:

$$A_{n+v} = \sum_h A_{n+v-\hat{\delta}_h} + \sum_{i,j} \left(\sum_{k=0}^{n+v-(\delta_i+\overline{\delta}_j)} \left(k + \frac{\delta_i' + \overline{\delta}_j'}{2} \right) d_k d_{n+v-(\delta_i+\overline{\delta}_j)-k} + \right.$$
$$\left. + \sum_{k=0}^{n+v-(\delta_i+\overline{\delta}_j)} A_k d_{n+v-(\delta_i+\overline{\delta}_j)-k} + \sum_{k=0}^{n+v-(\delta_i+\overline{\delta}_j)} A_{n+v-(\delta_i+\overline{\delta}_j)-k} d_k \right), \quad (6)$$

where the sum $\sum_h$ is intended for all $h = 1, \dots, \widehat{r}$ and the sum $\sum_{i,j}$ for all $i = 1, \dots, r$ and $j = 1, \cdots, \overline{r}$. From formula (6) we obtain a functional equation for the generating function $A(t)$ of any scheme R'. ■

Let us now apply the previous result to the scheme $R' = \{(1, 0, black),$ $(1, 1, black), (1, 2, black)\}$. In this case, we have $S = \{(1, 0, black)\}$, $\overline{S} = \{(1, 2, black)\}$, $\widehat{S} = \{(1, 1, black)\}$ and $v = 2$.

Theorem 3.2 *The total area determined by the Motzkin paths and the main diagonal is asymptotics to* $3^{n+1}/4$ *and therefore the corresponding average value is* $\sqrt{3\pi}/6n^{3/2}$.

Proof:
We can use formula (5) to obtain a functional equation for the paths number generating function $d(t)$:

$$d(t) = 1 + td(t) + t^2 d(t)^2.$$

The solution of this equation is:

$$d(t) = \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t^2},$$

which is known in literature as the generating function of *Motzkin numbers* M_n, whose asymptotic value is $M_n \sim 3^{n+1}\sqrt{3/\pi n}/2n$, (see Barcucci et al. [1]). Formula (6) gives us the following recurrence relation for the total area A_n :

$$A_{n+2} = A_{n+1} + \sum_{k=0}^{n} (k+1) d_k d_{n-k} + \sum_{k=0}^{n} A_k d_{n-k} + \sum_{k=0}^{n} A_{n-k} d_k.$$

Going on to generating functions, we note that $A_0 = A_1 = 0$, and obtain:

$$\frac{A(t)}{t^2} = \frac{A(t)}{t} + td(t)d'(t) + d(t)^2 + 2A(t)d(t),$$

or

$$A(t) = t^2 d(t) \frac{td'(t) + d(t)}{1 - t - 2t^2 d(t)}.$$

By substituting the expressions $d(t)$ and $d'(t)$, we finally get:

$$A(t) = \frac{(1 - t - \sqrt{1 - 2t - 3t^2})^2}{4t^2(1 - 2t - 3t^2)}.$$

The coefficient A_n of this function can be easily found as follows:

$$\begin{aligned} A_n = [t^n]A(t) = [t^n] \left(\frac{1}{2t^2} + \frac{1}{4(1+t)} + \frac{3}{4(1-3t)} + \right. \\ \left. - \frac{1}{2t^2\sqrt{1 - 2t - 3t^2}} + \frac{1}{2t\sqrt{1 - 2t - 3t^2}} \right) = \\ = \frac{(-1)^n}{4} + \frac{3^{n+1}}{4} - \frac{1}{2}(\beta_{n+2} - \beta_{n+1}), \end{aligned}$$

where $\beta_n = [t^n]1/\sqrt{1 - 2t - 3t^2}$ is the n^{th} *trinomial coefficient*, whose asymptotic value is $\beta_n \sim 3^n \sqrt{3/(\pi n)}/2$, (see, e.g. [1]). Thus, we have $A_n \sim 3^{n+1}/4$ and the average area A_n/M_n is asymptotic to $\sqrt{3\pi}/6n^{3/2}$. ■

We conclude our paper with an application of the first passage decomposition methodology to a path scheme belonging to the $\mathcal{R}$-model different from R'. Our example points out this approach's wide field of application in the evaluation of the average area determined by underdiagonal lattice paths.

Example 3.3 Let us examine the scheme $R = \{(1, 0, \mathit{black}), (0, 1, \mathit{black}), (1, 3, \mathit{black})\}$, and, as usual, let D be the set of the corresponding lattice paths. We denote the steps belonging to R by a, b, c, respectively. The previous examples allow us to determine the context-free grammar defining set D recursively:

$$D ::= \epsilon \mid aDbD \mid aDaDcD, \tag{7}$$

from which we get:

$$d(t) = 1 + td(t)^2 + t^3 d(t)^3.$$

By performing the decomposition indicated in formula (7), we find the relation:

$$\begin{aligned} A_{n+3} = 2\sum_{k=0}^{n+2} A_k d_{n+2-k} + \sum_{k=0}^{n+2} \left(k + \frac{1}{2}\right) d_k d_{n+2-k} + \\ +3\sum_{k=0}^{n}\sum_{j=0}^{k} A_j d_{k-j} d_{n-k} + \sum_{k=0}^{n}\sum_{j=0}^{n} (2k - j + 3)\, d_j d_{k-j} d_{n-k}. \end{aligned} \tag{8}$$

In this case, there are only two decompositions possible: the first, corresponds to the one shown in Figure 3, while the second is made of the following elements:

– an initial step of type a;

– a subpath in D translated one position to the right;
– a step of type a;
– a subpath in D translated two positions to the right;
– a step of type c;
– a subpath in D.

We note that the initial conditions for the area problem are $A_0 = 0, A_1 = 1/2, A_2 = 3$, and, in terms of generating functions, equation (8) becomes:

$$\frac{A(t) - t/2 - 3t^2}{t^3} = 2\frac{A(t)d(t) - t/2}{t^2} + \frac{td'(t)d(t) - t}{t^2} +$$
$$+\frac{d(t)^2 - 2t - 1}{2t^2} + 3d(t)^2A(t) + 3td'(t)d(t)^2 + 3d(t)^3,$$

which has a quite complicated solution.

In general, when the path scheme contains a large number of steps, then the corresponding generating function becomes complicated.

However, we are currently studying the problem of finding general recurrence relations for the total area of any scheme in the $\mathcal{R}$-model.

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