Asy2dim: a package for finding asymptotics for convolution matrices

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Abstract We present a method for obtaining asymptotics for the generic element of a two-dimensional convolution matrix which includes Stirling numbers of both kinds and some other interesting combinatorial quantities. Asy2dim is a computer algebra package which implement this method. The current version of Asy2dim is written in Maple V.3 [2].

Résumé Nous présentons une méthode pour calculer l'évaluation asymptotique d'un générique élément d'une matrice de convolution à deux dimensions. Cette classe d'éléments comprend, avec plusieurs quantités combinatoires, les nombres de Stirling de première et deuxième espèce. Asy2dim est un logiciel, developpé en Maple V.3 [2], qui réalise cette méthode.

1 Introduction

In many aspects of mathematics and computer science, e.g., combinatorics and algorithm analysis, the problem of finding an expression for the generic element of a sequence $\{f_k\}_{k\in N}$ come up. It is not always possible to find a closed form for the k^{th} element of the sequence and so we are satisfied to obtain an asymptotic estimate $A(k) \approx f_k$. A great number of techniques deal with this problem, many of which are based on the knowledge of the sequence's generating function $f(z) = \mathcal{G}\{f_k\}_{k\in N}$. In [7] we examine a method derived from Darboux's method and the implicit function theory which can be applied to a large number of infinite, low triangular arrays known as *convolution matrices*. This method allows us to obtain the asymptotic estimate for Stirling numbers and many other quantities of combinatorial interest. Computer algebra systems help us to execute the amount of computations usually required by this method. In fact, since the computations involved are symbolic in nature, symbolic computer algebra is the most appropriate tool for this method, especially if a specialized package is available for this purpose. The Maple package Asy2dim provides us with this. This short paper begins with a brief description of the method, (for a more detailed description the reader is referred to [7]). The rest of the paper consists in the description of the Maple package Asy2dim and its ability to evaluate the asymptotic value for the generic element of a two-dimensional convolution matrix.

2 The method

Let us consider the function $F(z)$, analytic in a neighborhood of 0, such that $F(0) = 1$. According to Carlitz [1] and Knuth [4], if we set

$$F_n(x) = [z^n]F(z)^x \qquad \text{and} \qquad f_{n,k} = [x^k]n!F_n(x),$$

then the array $\{f_{n,k}\}_{n,k\in N}$ is known as a *convolution matrix*. For a fixed ratio $p = n/k$ we define $m = n - k$, with $k \neq 0$ and $n \neq k$, and then we have:

$$f_{n,k} = \frac{n!}{k!}[z^m]\left(\frac{\ln F(z)}{z}\right)^{\frac{m}{p-1}}.$$

By the Lagrange Inversion formula (see, e.g., Goulden and Jackson [3]) and by setting:

$$\phi(z) = ((\ln F(z))/z)^{1/(p-1)},$$

a unique analytic function $w = w(z)$ exists having $w(0) = 0$ such that $w = z\phi(w)$ and

$$f_{n,k} = \frac{n!}{k!}[z^m]\frac{1}{1 - z\phi'(w)}.$$

Our method consists in obtaining the asymptotic development of the function $(1-z\phi'(w))^{-1}$ from which we deduce the asymptotic estimate of $f_{n,k}$. According to Darboux's method, the asymptotic development of this function depends on its dominating singularities $r_i (i = 1, 2, \ldots)$, i.e., on the singularities of minimal modulus. Fortunately, in the present case, it is possible to show that these singularities are the same as for $w = w(z)$, and can be found by means of the implicit functions theory (see Sprugnoli and Verri [8], for details). According to this theory, the singularities of $w = w(z)$ lying on the convergence circle are among the solutions to the system:

$$\begin{cases} G(z, w) = w - z\phi(w) = 0 \\ G'_w(z, w) = 1 - z\phi'(w) = 0. \end{cases}$$

We now assume that there is only one dominating singularity r such that $w(r) = s$; if there is a finite number $r_1, r_2, \ldots, r_k$ of singularities having the same minimal modulus, we can find the asymptotic development relative to each, and then sum up all the partial results. It can be proved that the following relations hold:

$$s = \frac{\phi(s)}{\phi'(s)} \qquad r = \frac{s}{\phi(s)} = \frac{1}{\phi'(s)}.$$

A result which assures the existence of a unique singularity r is due to Meir and Moon [5]; it requires that the coefficients of $F(z)$ be strictly positive.

The asymptotic formulas obtained by the application of this method holds for constant p and $n \to \infty$, but, as pointed out by Danièle Gardy [6], the range of application is much wider than indicated, in particular, the formulas are valid without restriction on $p = n/k$, as long as $n = k + O(k)$ and $m = n - k \to \infty$.

The main steps of our method are the following:

1. We determine the dominating singularity of $w = w(z)$ by first solving the *basic relation* $s = \phi(s)/\phi'(s)$ and then by computing $r = s/\phi(s) = 1/\phi'(s)$.

2. We determine the asymptotic development of $w = w(z)$ around its dominating singularity r. According to the Weirstrass preparation theorem, in a neighborhood of (r, s), $w(z)$ can be expanded into a series as follows:

$$w(z) = s + a_1 \left(1 - \frac{z}{r}\right)^{1/m} + a_2 \left(1 - \frac{z}{r}\right)^{2/m} + \dots$$

   The value of m is determined by the fact that

$$\frac{\partial G(r,s)}{\partial w} = 0, \frac{\partial^2 G(r,s)}{\partial w^2} = 0, \dots, \frac{\partial^{m-1} G(r,s)}{\partial w^{m-1}} = 0 \quad \text{and} \quad \frac{\partial^m G(r,s)}{\partial w^m} \neq 0$$

   The most common case is $m = 2$, and even though our next developments are limited to this value, there are not conceptual differences for obtaining similar results when $m \neq 2$. If we set $A = (1 - z/r)$ and m is equal to 2, then $w = w(z)$ has an asymptotic development around $z = r$, as follows:

$$w(z) = s + a_1\sqrt{A} + a_2 A + a_3\sqrt{A^3} + \dots$$

   We can obtain the coefficents a_i by applying the relation $w = z\phi(w)$ and equating like coefficients.

3. We use the asymptotic expression for $w(z)$ to compute the asymptotic development for $(1 - z\phi'(w))^{-1}$ and obtain:

$$\frac{1}{1 - z\phi'(w)} = c_{-1}\frac{1}{\sqrt{A}} + c_0 + c_1\sqrt{A} + \dots.$$

   From this expression, we deduce the asymptotic development for $f_{n,k}$.

3 The Asy2dim package

The Asy2dim package provides tools for:

- computing the basic relation involving $p = n/k$ and s;
- plotting the function $p = p(s)$ computed in the previous step;

- computing the first rows of a convolution matrix;
- computing the asymptotic approximation for the generic element $f_{n,k}$ of the convolution matrix;
- computing the numeric approximation for a specific element $f_{n,k}$.

The basic idea of the package is that every step in a series of manipulations should be controlled by the user. It is the user who has to tell the package Asy2dim which command has to be applied next. In order to help the user to decide this, an on-line help for each procedure is provided with a detailed description and examples.
In the following, we describe the main features of the package and give an example on how to proceed. The displayed output of the example corresponds to the outputs of the package under the SPARCstation SunOS version of Maple V.3; however Asy2dim can be used on every computer with Maple V.3 installed.

4 Working example: Stirling numbers of the second kind

This example is about Stirling numbers of the second kind $f_{n,k} = \left\{ {n \atop k} \right\}$ and represents the simplest case of specific interest. For these numbers, we have $F(z) = \exp(e^z - 1)$.

> read Asy2dim:
> F:=exp(exp(z)-1);

$$F := \mathrm{e}^{(\mathrm{e}^z - 1)}$$

The `Mconv(F,z,R,C)` procedure computes the first `R` rows and `C` columns of the convolution matrix corresponding to the function `F` of the variable `z`.

> Mconv(F,z,7,7);

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}$$

The `asyconv(F,z,n,k,1)` procedure computes the first approximation for the coefficient, indexed `n`, `k`, of the convolution matrix.

```
> asyconv(F,z,n,k,1);
```

$$-\frac{1}{2}n!\,(\,e^{\%1}\,\%1-e^{\%1}+1\,)\,\%1\,(\,-n+k\,)\,(\,e^{\%1}-1\,)\,\sqrt{2}\,(\,-1\,)^{(\,n-k\,)}\text{binomial}(\,2\,n-2\,k,n-k\,)$$
$$\left(\frac{\left(\dfrac{e^{\%1}-1}{\%1}\right)^{\left(-\frac{k}{-n+k}\right)}k\,(\,e^{\%1}\,\%1-e^{\%1}+1\,)}{\%1\,(\,-n+k\,)\,(\,e^{\%1}-1\,)}\right)^{(\,n-k\,)}\Bigg/\Big(4^{(\,n-k\,)}\Big((\,-n+k\,)^2$$
$$\%1^2\,(\,e^{\%1}-1\,)^2\Big/\Big(k\,\Big(k\,(\,e^{\%1}\,)^2\,\%1^2-2\,e^{\%1}\,n+(\,e^{\%1}\,)^2\,n+e^{\%1}\,\%1^2\,k$$
$$+n-2\,k\,(\,e^{\%1}\,)^2\,\%1+2\,k\,e^{\%1}\,\%1-e^{\%1}\,\%1^2\,n\Big)\Big)\Big)^{1/2}\Big(k\,(\,e^{\%1}\,)^2\,\%1^2$$
$$-2\,e^{\%1}\,n+(\,e^{\%1}\,)^2\,n+e^{\%1}\,\%1^2\,k+n-2\,k\,(\,e^{\%1}\,)^2\,\%1+2\,k\,e^{\%1}\,\%1$$
$$-e^{\%1}\,\%1^2\,n\Big)\,k!\Big)$$
$$\%1 := \text{RootOf}(\,\_Z\,(\,k\,e^{\_Z}\,\_Z-n\,e^{\_Z}+n\,)\,)$$

The solution is expressed in terms of a functional equation whose solution is the singularity s. Since s is the solution to the functional equation $k\exp(s)s-n\exp(s)+n=0$, we can eliminate every exponential term by applying the following substitution:

```
> substitution:=subs(`Z=s,exp(`Z)=solve(op(%1),exp(`Z)));
```

$$\mathit{substitution} := e^s = -\frac{n}{k\,s-n}$$

By expressing e^s in rational terms, we obtain:

```
> AStir2:=unapply(normal(subs(%1=s,substitution,"")),n,k,s);
```

$$\mathit{AStir2} := (\,n,k,s\,)\to-\frac{1}{2}n!\,s\,(\,-n+k\,)\,k\,\sqrt{2}\,(\,-1\,)^{(\,n-k\,)}\text{binomial}(\,2\,n-2\,k,n-k\,)$$
$$\left(-\frac{\left(-\dfrac{k}{k\,s-n}\right)^{\left(-\frac{k}{-n+k}\right)}}{s}\right)^{(\,n-k\,)}\Bigg/\left(k!(\,k\,s+k-n\,)\,n\,\sqrt{-\frac{(\,-n+k\,)\,s^2\,k}{(\,k\,s+k-n\,)\,n}}\,4^{(\,n-k\,)}\right)$$

This is the formal approximation desired for Stirling numbers of the second kind. We can now compute some numerical values, and compare them with the true values:

```
> for j from 5 to 45 by 10 do
>     evalf(combinat[stirling2](50,j)),evalf(AStir2(50,j,sing(F,z,50,j)))
> od;
```

$$.7400958644\,10^{33},\,.7393232349\,10^{33}$$

$$.2922645700\,10^{47}, .2922697575\,10^{47}$$

$$.7453802153\,10^{43}, .7448768232\,10^{43}$$

$$.2423553632\,10^{32}, .2418200655\,10^{32}$$

$$.1313282929\,10^{14}, .1302637868\,10^{14}$$

The `sing(F,z,n,k)` procedure computes the value s, related to the singularity r of the function $(1 - z\phi'(w))^{-1}$ with $p = n/k$, by solving the basic relation $s = \phi(s)/\phi'(s)$. It is sometimes necessary to specify a fifth parameter indicating a range containing s and so we propose the `drawsing(F,z)` procedure which gives the plot of the function $p = p(s)$ deduced from the basic relation.

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