Algebraic and combinatorial properties of simple, coloured walks

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Abstract. We investigate the algebraic rules for functionally inverting a Riordan array given by means of two analytic functions. In this way, we find an extension of the Lagrange Inversion Formula and we apply it to some combinatorial problems on simple coloured walks. For some of these problems we give both an algebraic and a combinatorial proof.

1 Introduction

Shapiro et al. [15] introduced the concept of a *Riordan array* in 1991. Their aim was to generalize the concept of *renewal array* as defined by Rogers [11] as far back as 1978, in an attempt to define a class of infinite low triangular arrays having properties similar to those of the Pascal triangle. As a matter of fact, similar concepts had been introduced in the literature several years earlier. For example, Hoggatt and Bicknell published a series of papers in the "Fibonacci Quarterly" dealing with what they called *convolutory array* or *convolution triangles* (see, e.g., [1, 6]), which are actually renewal arrays. Even before that, Jabotinsky [7, 8] had introduced an analogous concept, later "re-invented" by Carlitz [2]. This concept was used in the theory of Umbral Calculus [12, 13] and, more recently, it was discussed by Knuth [9] who called it *convolution matrices*. The only difference between this concept and renewal arrays is that it uses exponential generating functions instead of ordinary ones. When dealing with combinatorial sums, Riordan arrays seem to be a very powerful tool (see Theorem 2.1 below and Sprugnoli [16]). Furthermore, it is obvious that every convolution array $\{f_{n,k}\}_{n,k\in N}$ can be trasformed into an equivalent Riordan array when we consider the array $\{\frac{k!}{n!}f_{n,k}\}_{n,k\in N}$.

In the present paper we consider the class of simple coloured walks, i.e., walks on the integral lattice Z^2 made up of unitary steps (hence we call them *simple*) in three directions: east, north-east, north. Furthermore, we can have different kinds of steps in each direction, and each kind is conventionally distinguished by a different colour. Some well-known combinatorial structures, such as binary trees, Motzkin trees and parenthesized expressions, correspond to some particular cases of simple coloured walks. Coloured walks are also a way for dealing with simple walks in which the three kinds of steps can occur with different probabilities.

We show that simple coloured walks correspond to some particular Riordan arrays, and that therefore many combinatorial problems concerning these walks can be algebraically solved by using the corresponding Riordan array and the relative method of performing sums. Many of these problems are very difficult from a combinatorial point of view, so we have tried to give bijective proofs as well, at least for the simplest cases.

In Section 2, we summarize the main properties of Riordan arrays used in our paper: the explicit form of an element in the inverse of a Riordan array is new, although some similar properties are present in the Umbral Calculus theory. In Section 3, we define various kinds of simple coloured walks and prove some basic properties for them. Finally, in Section 4, we give the algebraic proof of many complicated identities on simple coloured walks and a combinatorial proof of some of them in order to evidence the difference between the two approaches.

2 Riordan Arrays

Let $\{f_k\}_{k\in N}$ be a sequence of (real) numbers. The generating function $f(t)$ of the sequence is defined as $f(t) = \mathcal{G}_t\{f_k\}_{k\in N} = \sum_{k=0}^{\infty} f_k t^k$ (the subscript t in $\mathcal{G}_t$ binding the indeterminate t, will almost always be understood). Moreover, given an analytic function $f(t)$, the notation $[t^n]f(t)$ will denote the coefficient of t^n in the Taylor development of $f(t)$ around $t = 0$. The same notation will be applied to formal power series. A *Riordan array* is an infinite low triangular array $\{\, d_{n,k} \,|\, n, k \in N\}$ defined by a couple $(d(t), h(t))$ of formal power series in the sense that $d_{n,k} = [t^n]d(t)(th(t))^k$. As we shall see, $(d(t), h(t))$ is a sort of "generating function" that defines the Riordan array. We are interested in extracting the coefficient $[t^n]$ from some generating function depending on $d(t)$ and $h(t)$. When we are able to find the exact value of the coefficient, $d(t)$ and $h(t)$ can be considered as general formal power series. However, we are sometimes unable to extract the coefficient and may be satisfied with an asymptotic value. In this case, we should use analytic considerations and therefore require that $d(t)$ and $h(t)$ be analytic functions. We usually assume that $d(0) \neq 0$; when we also have $h(0) \neq 0$, the Riordan array is called *proper*. Another way of characterizing Riordan arrays is to consider their corresponding bivariate generating functions. We can easily prove that:

$$d(t, w) = \frac{d(t)}{1 - twh(t)}.$$

The fundamental property of Riordan arrays is given by the following result:

Theorem 2.1 *Let $D = (d(t), h(t))$ be a Riordan array and let $f(t)$ be the generating function of the sequence $\{f_n\}_{n\in N}$. Then:*

$$\sum_{k=0}^{\infty} d_{n,k} f_k = [t^n]d(t)f(th(t)).$$

Conversly, if $\{\, d_{n,k} \,|\, n, k \in N\}$ is an infinite triangle such that for every sequence $\{f_k\}_{k\in N}$ we have:

$$\mathcal{G}\left\{\sum_{k=0}^{\infty} d_{n,k} f_k\right\} = d(t)f(th(t)),$$

where $f(t)$ is the generating function of the sequence f_k, and $d(t)$, $h(t)$ are two analytic functions not depending on $f(t)$, then the triangle defined by the Riordan array $D = (d(t), h(t))$ coincides with $\{d_{n,k}\}$.

The reader is referred to Sprugnoli [16] for the proof. Proper Riordan arrays are further characterized by the following property, proved in Rogers [11].

Theorem 2.2 *Let $(d(t), h(t))$ be a proper Riordan array; then a sequence $A = \{a_k\}_{k\in N}$ exists such that $\forall n \in N$*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \dots$$

and the generating function $A(t)$ of the sequence A is uniquely determined by the relation $h(t) = A(th(t))$.

The usual row-by-column product of two Riordan arrays is a Riordan array. This can be proved in the following way; if $D = (d(t), h(t)) = \{d_{n,k}\}$ and $F = (f(t), g(t)) = \{f_{n,k}\}$ then:

$$D * F = \{\sum_j d_{n,j} f_{j,k}\} = \{\sum_j [t^n] d(t)(th(t))^j [y^j] f(y)(yg(y))^k\} =$$

$$= \{[t^n] d(t) \sum_j ([y^j] f(y)(yg(y))^k)(th(t))^j\} =$$

$$= \{[t^n] d(t) f(th(t))(th(t)g(th(t)))^k\} = (d(t)f(th(t)), h(t)g(th(t))).$$

Obviously, the Riordan array $I = (1, 1)$ is the only identity. So, if $D = (d(t), h(t))$, is a proper Riordan array, it has an inverse $D^{-1} = (\bar{d}(t), \bar{h}(t))$

$$(d(t), h(t)) * (\bar{d}(t), \bar{h}(t)) = (d(t)\bar{d}(th(t)), h(t)\bar{h}(th(t))) = (1, 1)$$

and therefore

$$\bar{d}(y) = \left[\frac{1}{d(t)} \middle| y = th(t)\right], \qquad \bar{h}(y) = \left[\frac{1}{h(t)} \middle| y = th(t)\right].$$

The existence and uniqueness of $\bar{d}(t)$, $\bar{h}(t)$ is guaranteed by the Lagrange Inversion Formula (LIF). There is a striking relationship between the function $\bar{h}(t)$ and the A-sequence for D, and, conversely, between $h(t)$ and the A-sequence for D^{-1}. By Theorem 2.2 we have:

$$\bar{h}(y) = \bar{h}(th(t)) = \frac{1}{h(t)} = \frac{1}{A(th(t))} = \frac{1}{A(y)}.$$

In general, no simple relationship exists between $d(t)$, $\bar{d}(t)$ and the other quantities. A fine property of coloured walk arrays is that a simple connection can be found between all the functions involved in any array and their inverse.

The Lagrange Inversion Formula can be further used to find an explicit formula for the elements of D^{-1} as a function of the proper Riordan array $D = (d(t), h(t))$, as follows:

Theorem 2.3 *Let $D = (d(t), h(t))$ be any proper Riordan array. Then the elements $\bar{d}_{n,k}$ of the inverse Riordan array $D^{-1} = (\bar{d}(t), \bar{h}(t))$ are given by the formula*

$$\bar{d}_{n,k} = \frac{1}{n}[y^{n-k}]\left(k - \frac{yd'(y)}{d(y)}\right)\frac{1}{d(y)h(y)^n}.$$

Proof. We have

$$\bar{d}_{n,k} = [t^n][w^k]\frac{\bar{d}(t)}{1 - tw\bar{h}(t)} = [t^n][w^k]\left[\frac{\bar{d}(t)}{1 - wy}\middle| y = t\bar{h}(t)\right].$$

We now observe that $h(y)\bar{h}(yh(y)) = 1$ and $\bar{h}(t)h(t\bar{h}(t)) = 1$; consequently $y = t\bar{h}(t)$ if and only if $t = yh(y)$ and therefore:

$$\bar{d}_{n,k} = [w^k][t^n]\left[\frac{\bar{d}(t)}{1 - wy}\middle| t = yh(y)\right] = [w^k][t^n]\left[\frac{1}{d(y)(1 - wy)}\middle| y = th^{-1}(y)\right].$$

By applying the LIF, we now obtain:

$$\bar{d}_{n,k} = [w^k]\frac{1}{n}[y^{n-1}]\left(\frac{\partial}{\partial y}\frac{1}{d(y)(1 - wy)}\right)h^{-n}(y) =$$

$$= [w^k]\frac{1}{n}[y^{n-1}]\left(\sum_{r=0}^{\infty} w^{r+1}y^r(r+1) - \frac{d'(y)}{d(y)}\sum_{r=0}^{\infty} w^r y^r\right)\frac{1}{d(y)h(y)^n} =$$

$$= \frac{1}{n}[y^{n-k}]\left(k - \frac{yd'(y)}{d(y)}\right)\frac{1}{d(y)h(y)^n}.$$

■

Obviously, when $k = 0$, Theorem 2.2 reduces to the LIF. Moreover, whenever we find an identity like

$$\sum_{k=0}^{\infty} d_{n,k} f_k = g_n,$$

by Theorem 2.1, we can immediately conclude that the inverse identity

$$\sum_{k=0}^{\infty} \bar{d}_{n,k} g_k = f_n$$

also holds. See [4, 10] for some significant examples treated by other methods.

3 The Riordan Arrays for Coloured Walks

Let us consider the Z^2 lattice and the finite walks starting from the origin O and composed by:

i) *east* steps, i.e., steps from (x, y) to $(x+1, y)$; these steps can be distinguished by their colour, which is chosen from a set of a different colours;
ii) *north-east* steps, i.e., steps from (x, y) to $(x + 1, y + 1)$; these steps can be distinguished by their colour, which is chosen from a set of b different colours;
iii) *north* steps, i.e., steps from (x, y) to $(x, y+1)$; these steps can be distinguished by their colour, which is chosen from a set of c different colours.

Since each step is unitary, i.e., x and/or y are only incremented by 1, the walks are called *simple*; they are also called *coloured* because of the different colours each step can take on. We are interested in *weakly underdiagonal walks*, that is, walks whose ending point (x, y) is such that $x \geq y$; in other words, the ending point is not situated above the main diagonal $x = y$. When the walk never goes above this diagonal, we simply call it an *underdiagonal walk*. We are interested in simple coloured (weakly) underdiagonal walks and, for simplicity's sake, the term *walk* will henceforth refer to this kind of walk. In Table 1, we give the number of underdiagonal and weakly underdiagonal walks when $a = 1, b = 2, c = 1$, i.e., when there only is one colour for the east and north steps, but two colours (for ex., black and red) for the diagonal, or north-east, steps. In Table 1, n refers to the total number of steps which make up the walks and k refers to the distance of the ending point of the walk from the main diagonal; the distance is measured along the $x-$ or $y-$axis. So, for example, there are 48 underdiagonal walks of length 4 ending just one position below the main diagonal, while there are 56 analogous weakly underdiagonal walks. In Figure 1, we show the 5 underdiagonal walks corresponding to $n = 2$, $k = 0$, and, on the right, we indicate the only walk to be added to obtain the corresponding weakly underdiagonal walks. It

n/k	0	1	2	3	4	5
0	1					
1	2	1				
2	5	4	1			
3	14	14	6	1		
4	42	48	27	8	1	
5	132	165	110	44	10	1

n/k	0	1	2	3	4	5
0	1					
1	2	1				
2	6	4	1			
3	20	15	6	1		
4	70	56	28	8	1	
5	252	210	120	45	10	1

Table 1. Number of underdiagonal and weakly underdiagonal walks for $a = 1, b = 2, c = 1$

is a simple matter to prove the following:

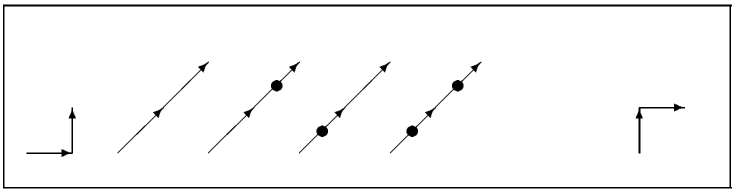


Fig. 1. Underdiagonal and weakly underdiagonal walks for $n = 2, k = 0$

Theorem 3.1 *The infinite arrays $\{p_{n,k}\}$ and $\{q_{n,k}\}$, where $p_{n,k}$ are the n-long underdiagonal walks ending at a distance k from the main diagonal, and $q_{n,k}$ are the corresponding weakly underdiagonal walks, are two Riordan arrays having (a, b, c) as their A-sequence.*

Proof. For $k > 0$, the walks counted by $p_{n,k}$ are made up of:

i) the walks of length $n-1$ ending at distance $k-1$ from the main diagonal, followed by any one of the a coloured east steps;
ii) the walks of length $n-1$ ending at distance k from the main diagonal, followed by any one of the b coloured north-east steps;
iii) the walks of length $n-1$ ending at distance $k+1$ from the main diagonal, followed by any one of the c coloured north steps.

Since the numbers in the column $k = 0$ are arbitrary, we are able to prove our statement by means of Theorem 2.2. ■

Theorem 2.2 also allows us to find the h-function, common to the two Riordan arrays. In fact, the relation $h(t) = A(th(t))$ becomes

$$h(t) = ct^2h(t)^2 + bth(t) + a$$

and the solution having $h(0) \neq 0$ is:

$$h(t) = \frac{1 - bt - \sqrt{\Delta}}{2ct^2},$$

where $\Delta = 1 - 2bt + (b^2 - 4ac)t^2$.

Theorem 3.2 *The explicit forms of the two Riordan arrays for the underdiagonal and weakly underdiagonal walks are, respectively:*

$$\{p_{n,k}\} = \left(\frac{1 - bt - \sqrt{\Delta}}{2act^2}, \frac{1 - bt - \sqrt{\Delta}}{2ct^2}\right), \qquad \{q_{n,k}\} = \left(\frac{1}{\sqrt{\Delta}}, \frac{1 - bt - \sqrt{\Delta}}{2ct^2}\right).$$

Proof. A reasoning analogous to the one in the proof of Theorem 3.1 allows us to express the generating function for the sequence $\{p_{n+1,0}\}_{n\in N}$ in terms of $h(t)$, as follows

$$\frac{d(t)-1}{t} = bd(t) + ctd(t)h(t).$$

This equation can be solved with respect to $d(t)$ by giving the d-function for the underdiagonal walks. As far as weakly underdiagonal walks are concerned, let us first consider the ones corresponding to the A-sequence (c,b,a); the discriminant Δ is the same as before and therefore the h-function is $\tilde{h}(t) = (1-bt-\sqrt{\Delta})/(2at^2)$. We can now observe that $q_{n+1,0}$ is obtained by summing b times $q_{n,0}$, c times the column $k=1$ and a times the column $k=1$ in the (c,b,a) case, which is column $k=-1$ in our case. Consequently, we get the following relation:

$$\frac{d(t)-1}{t} = bd(t) + ctd(t)h(t) + atd(t)\tilde{h}(t).$$

By using the expression found for $\tilde{h}(t)$ and solving for $d(t)$, we find the formula in the statement of the Theorem. ∎

Another interesting kind of walks are *Shapiro's walks* [14], which are characterized by the fact that the north-east steps on the main diagonal may have different colours from the north-east steps on the other diagonals. In other words, we have (a,b,c) colours as in the walks above, but on the main diagonal b' colours ($b'\neq b$) are used for north-east steps. In Table 2, we illustrate the triangles for Shapiro's underdiagonal and weakly underdiagonal walks when $a=1, b=2, c=1$ (as in Table 1), but $b'=1$. We can apply Theorem 3.2's proof to Shapiro's arrays:

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	2	3	1			
3	5	9	5	1		
4	14	28	20	7	1	
5	42	90	75	35	9	1

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	3	3	1			
3	9	10	5	1		
4	29	34	21	7	1	
5	97	117	83	36	9	1

Table 2. Number of Shapiro's underdiagonal and weakly underdiagonal walks for $a=1, b=2, c=1, b'=1$

Theorem 3.3 *The explicit forms of the two Riordan arrays of underdiagonal and weakly underdiagonal Shapiro's walks are, respectively:*

$$\{s_{n,k}\} = \left(\frac{1-(2b'-b)t-\sqrt{\Delta}}{2t(b-b'+(ac+b'^2-bb')t)}, \frac{1-bt-\sqrt{\Delta}}{2ct^2}\right)$$

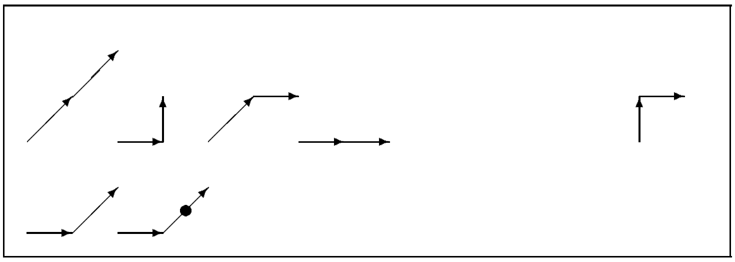


Fig. 2. Shapiro's underdiagonal and weakly underdiagonal walks for $n = 2$

$$\{r_{n,k}\} = \left(\frac{(b-b')t - \sqrt{\Delta}}{t(2b + t(b'^2 - 2bb' + 4ac)) - 1}, \frac{1 - bt - \sqrt{\Delta}}{2ct^2} \right).$$

Proof. The function $h(t)$ is only determined by the three values a, b, c and therefore coincides with the previous case. To compute the function $d(t)$, we observe that $s_{n+1,0} = b' s_{n,0} + c s_{n,1}$. Therefore, we have the relation

$$\frac{d(t) - 1}{t} = b'd(t) + cth(t)d(t).$$

By solving for $d(t)$, we obtain our result. In a similar way, we obtain the function $d(t)$ for weakly underdiagonal walks. ■

From an algebraic point of view, it is important to determine the Riordan arrays which are the inverse of the above walk arrays. As we observed at the end of Section 2, every combinatorial sum $\sum_{k=0}^{\infty} d_{n,k} f_k = g_n$ involving the number of (weakly) underdiagonal walks can be trasformed into another combinatorial sum $\sum_{k=0}^{\infty} \bar{d}_{n,k} g_k = f_n$. Moreover, as will be shown further on, the inverse arrays can be useful for finding formulas relative to the single elements of the original arrays. Unfortunately, there is no simple way (as far as we know) to attach a combinatorial meaning to the numbers appearing in the inverse arrays.

Theorem 3.4 *The inverse of the four kinds of walk arrays considered above are:*

$$\{\bar{p}_{n,k}\} = \left(\frac{a}{a + bt + ct^2}, \frac{1}{a + bt + ct^2} \right), \quad \{\bar{q}_{n,k}\} = \left(\frac{a - ct^2}{a + bt + ct^2}, \frac{1}{a + bt + ct^2} \right),$$

$$\{\bar{s}_{n,k}\} = \left(\frac{a + (b-b')t}{a + bt + ct^2}, \frac{1}{a + bt + ct^2} \right),$$

$$\{\bar{r}_{n,k}\} = \left(\frac{a + (b-b')t - ct^2}{a + bt + ct^2}, \frac{1}{a + bt + ct^2} \right).$$

Proof. As a consequence of Theorem 2.2, we have:

$$\bar{h}(t) = \frac{1}{A(t)} = \frac{1}{a + bt + ct^2},$$

which is the h-function common to the four cases. Let us now consider the underdiagonal walks $\{p_{n,k}\}$:

$$\bar{d}(t) = \left[\frac{1}{d(y)} \middle| \, t = yh(y) \right].$$

We can obtain an explicit value for $y = y(t)$ having $y(0) = 0$ because the relation between y and t is a second order equation. Thus, we find:

$$y = \frac{t}{a + bt + ct^2},$$

and by substituting this value in $1/d(y)$ we immediately obtain the result desired. The other expressions are found in the same way. ∎

In a sense, these inverse arrays are simpler than the original ones and allow us to apply Theorem 2.3 to look for an explicit form relative to the number of walks. The simplest case is given by the underdiagonal walks.

Theorem 3.5 *The number $\{p_{n,k}\}$ of n-length underdiagonal walks ending at a distance k from the main diagonal is:*

$$p_{n,k} = \frac{k}{na}[y^{n-k}](a + by + cy^2)^{n+1} + \frac{1}{na}[y^{n-k-1}](b + 2cy)(a + by + cy^2)^n.$$

Proof. By applying Theorem 2.3, we immediately obtain:

$$p_{n,k} = \frac{1}{n}[y^{n-k}] \left(k - \frac{y\bar{d}'(y)}{\bar{d}(y)} \right) \frac{1}{\bar{d}(y)\bar{h}(y)^n} =$$

$$= \frac{1}{n}[y^{n-k}] \left(k + y\frac{b + 2cy}{a + by + cy^2} \right) \frac{a + by + cy^2}{a}(a + by + cy^2)^n.$$

Then, by developing this expression, we easily obtain the result desired. ∎

Obtaining the result desired for other kinds of walks is not difficult from a conceptual point of view but it requires several computations, which the reader can easily do by himself.

When $b^2 - 4ac = 0$, the trinomial $a + bt + ct^2$ can be expressed as the square of a binomial, and we get the following result:

Theorem 3.6 *For the underdiagonal walks corresponding to a, b, c, such that $b^2 - 4ac = 0$, we have the following explicit formulas:*

$$p_{n,k} = a^n \frac{k+1}{n+1} \left(\frac{2c}{b}\right)^{n-k} \binom{2n+2}{n-k}$$

$$\bar{p}_{n,k} = \left(\frac{2c}{b}\right)^{n-k} \frac{1}{a^k} \binom{n+k+1}{n-k} (-1)^{n-k}.$$

Proof. The formula of Theorem 3.5 reduces to:

$$p_{n,k} = \frac{k}{na}[y^{n-k}]a^n(\frac{2c}{b}y+1)^{2n+2} + \frac{1}{na}[y^{n-k-1}](b+2cy)a^n(\frac{2c}{b}y+1)^{2n} =$$

$$= \frac{a^n}{n}\left[[y^{n-k}]k(\frac{2c}{b}y+1)^{2n+2} + [y^{n-k-1}](b+2cy)\frac{4c}{b}(\frac{2c}{b}y+1)^{2n+1}\right] =$$

$$= \frac{a^n}{n}\left[k\binom{2n+2}{n-k}\left(\frac{2c}{b}\right)^{n-k} + \frac{4c}{b}\binom{2n+1}{n-k-1}\left(\frac{2c}{b}\right)^{n-k-1}\right].$$

By developing this expression, we easily get the first result desired and from the Riordan array definition, we obtain:

$$\bar{p}_{n,k} = [t^n]\frac{at^k}{a^{k+1}\left(\frac{2c}{b}t+1\right)^{2k+2}} = \frac{1}{a^k}\binom{-2k-2}{n-k}\left(\frac{2c}{b}\right)^{n-k}$$

and this proves the second result. ∎

4 Identities on symmetric walks

In this section, we investigate *symmetric* walks, i. e., walks having $A = (a, b, a)$ as their $A-$sequence. Symmetric coloured walks are characterized by a large number of combinatorial identities. The concept of Riordan arrays allows us to give almost immediate proofs of these identities. However, when we also try to derive a combinatorial proof, we frequently encounter difficulties and have rarely obtained positive results. Consequently, the aim of this section is to give several examples of combinatorial identities for which we can provide both an algebraic and a combinatorial proof, and to illustrate some other examples for which only algebraic proofs are available.

Let us denote by $q_{n,k}^{[a,b]}$ the number of symmetric weakly underdiagonal walks of length n ending at a distance k from the main diagonal, by $p_{n,k}^{[a,b]}$ and $s_{n,k}^{[a,b,b']}$ the corresponding underdiagonal walks and Shapiro's underdiagonal walks, and by $\beta^{[a,b]}$, $\mu^{[a,b]}$ and $\sigma^{[a,b,b']}$ the following functions:

$$\beta^{[a,b]}(t) = \frac{1}{\sqrt{\Delta}}, \qquad \mu^{[a,b]}(t) = \frac{1-bt-\sqrt{\Delta}}{2at^2}$$

and

$$\sigma^{[a,b,b']}(t) = \frac{1-(2b'-b)t-\sqrt{\Delta}}{t(b-b'+(ac+b'^2-bb')t)},$$

where, $\Delta = 1-2bt+(b^2-4a^2)t^2$. According to Theorems 3.2 and 3.3, the Riordan arrays associated to the walks above are

$$\left\{q_{n,k}^{[a,b]}\right\} = \left(\beta^{[a,b]}(t), \mu^{[a,b]}(t)\right), \qquad \left\{p_{n,k}^{[a,b]}\right\} = \left(\frac{\mu^{[a,b]}(t)}{a}, \mu^{[a,b]}(t)\right)$$

and

$$\left\{s_{n,k}^{[a,b,b']}\right\} = \left(\sigma^{[a,b,b']}(t), \mu^{[a,b]}(t)\right).$$

By using the Darboux method [3, 5], we can evaluate the following asymptotic approximation:

Theorem 4.1 *The total number of symmetric weakly underdiagonal walks is approximated by:*

$$\sum_{k=0}^{n} q_{n,k}^{[a,b]} \approx \frac{1}{2}(b+2a)^n + \sqrt{\frac{b+2a}{a}}\binom{2n}{n}\frac{(b+2a)^n}{4^{n+1}}\left(\frac{b-2a}{8a(2n-1)}-1\right).$$

We can now prove some relationships among the elements of the above-mentioned families of walks having both an algebraic and a combinatorial proof:

Theorem 4.2 *We have:*

$$q_{n,k}^{[a,b+h]} = \sum_{i=0}^{n}\binom{n}{i} h^{n-i} q_{i,k}^{[a,b]}, \qquad p_{n,k}^{[a,b+h]} = \sum_{i=0}^{n}\binom{n}{i} h^{n-i} p_{i,k}^{[a,b]}.$$

Proof. By applying Theorem 2.1 to the Pascal triangle, we get:

$$\sum_{i=0}^{\infty}\binom{n}{i} f_i = [t^n]\frac{1}{1-t} f\left(\frac{t}{1-t}\right) = [t^n]\Omega(f(t))$$

and by iterating the application, we obtain:

$$\sum_{i=0}^{\infty}\binom{n}{i} h^{n-i} f_i = [t^n]\frac{1}{1-ht} f\left(\frac{t}{1-ht}\right) = [t^n]\Omega^h(f(t)).$$

From the previous relation we have:

$$\Omega^h(\beta^{[a,b]}(t)) = \beta^{[a,b+h]}(t) \qquad \text{and} \qquad \Omega^h(\mu^{[a,b]}(t)) = \mu^{[a,b+h]}(t);$$

therefore, for underdiagonal walks, we obtain:

$$q_{n,k}^{[a,b+h]} = [t^n]\beta^{[a,b+h]}(t)(t\mu^{[a,b+h]}(t))^k = [t^n]\Omega^h(\beta^{[a,b]}(t)(t\mu^{[a,b]}(t))^k) =$$

$$= \sum_{i=0}^{n}\binom{n}{i} h^{n-i} q_{i,k}^{[a,b]}.$$

The combinatorial explanation trivially follows by observing that the coefficient $\binom{n}{i} h^{n-i}$ represents the number of ways to insert $n-i$ north-east steps with h colours into an i-length walk. The corresponding identity for weakly underdiagonal walks is found in the same way. ∎

Theorem 4.3 *If* $\{f_k\} = \{1, 0, -1, 0, 1, 0, -1, 0, 1, 0, \ldots\}$, *then*

$$\sum_{k=0}^{n} p_{n,k}^{[a,b]} f_k = b^n.$$

Proof. Let us start with the algebraic proof. The generating function of the sequence f_k is:

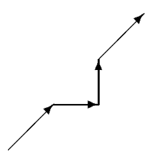
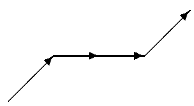
$$f(t) = \frac{1}{1+t^2}.$$

Then, by Theorem 2.1, we obtain:

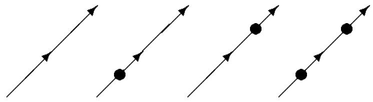
$$\sum_{k=0}^{n} p_{n,k}^{[a,b]} f_k = [t^n]\frac{1}{a}\mu^{[a,b]}(t) f(t\mu^{[a,b]}(t)) =$$

$$= [t^n]\frac{1-bt-\sqrt{\Delta}}{2a^2t^2} f\left(\frac{1-bt-\sqrt{\Delta}}{2at}\right) = [t^n]\frac{1}{1-bt} = b^n.$$

Now let us discuss the combinatorial interpretation: if we multiply the numbers $p_{n,k}^{[a,b]}$ by the sequence f_k, we exclude the walks ending at an odd distance from the main diagonal because $f_{2k+1} = 0$; we assign *weight* $+1$ or -1 to the walks ending at an even distance. The theorem asserts that this weighted sum has value b^n, which corresponds to the number of walks made up of a north-east step only. There is a bijection between the walks ending at distance k whose last non-north-east step is north, and the walks ending at distance $k+2$ whose last non-north-east step is east. By transforming the last north step into an east step, the distance from the main diagonal increases by two. This bijection does not concern the b^n walks made up of north-east steps only and, therefore, if we give weight $+1$ and -1 alternatively to the sets in bijection, we get the proof. For example, if $A = (1, 1, 1)$, $n = 4$ and $k = 0$, the walk

 corresponds to the walk 

having $n = 4$ and $k = 2$. The b^n excluded walks made up of north-east steps only are the following if $A = (1, 2, 1)$ and $n = 2$:



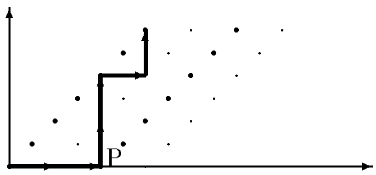
∎

The following identity concerns the underdiagonal walks not having north-east steps and has both an algebraic and a combinatorial proof.

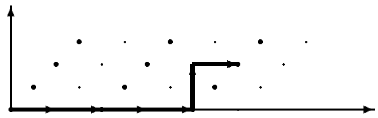
Theorem 4.4 *If* $\{f_k\} = \{1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, \ldots\}$, *then*

$$\sum_{k=0}^{n} p_{n,k}^{[a,0]} f_k = a^n.$$

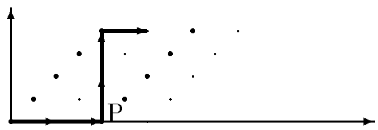
Proof. The algebraic proof is the same as the one given in Theorem 4.3, with $f(t) = (1-t+t^2)^{-1}$. The combinatorial proof, however, is more complicated. We only take into consideration walks made up of east and north steps: obviously they exist only if n and k are both even or both odd. If we multiply the number of these walks by the sequence f_k, we only consider the walks ending at a distance $k \neq 2+3i$ from the main diagonal, for some positive integer i because $f_{2+3i} = 0$. In particular, if n is even (odd), the value of k must be even (odd), too, and different from $2+3i$. We define as *even*, diagonals at an even distance from the main diagonal and *odd*, the ones at an odd distance; once again the distance is measured along the $x-$ or $y-$axis. According to this definition, the main diagonal is even. Let us first examine an even integer n. Our aim is to establish a bijection between the walks ending at distance k from the main diagonal and either the walks ending at distance $k+2$ if $k \neq 3i$ or the ones ending at distance $k+4$ if $k = 3i$. In the first case, if we want to transform a walk ending at distance k into a walk ending at distance $k+2$, we need to move its ending point from the even diagonal k to the following even diagonal $k+2$. In order to do this, we take the point P having the highest coordinates among the points of the walk that intersect the odd diagonal $k+1$. Then we consider the walk made up of the same steps up to P and we add the remaining piece of walk specular with respect to the odd diagonal. The walk obtained is the corresponding walk ending at distance $k+2$. In the second case, we consider the point P having the highest coordinates among the points of the walk that intersect the even diagonal $k+2$. Then we take the walk made up of the same steps up to P and we add the remaining piece of walk specular with respect to the even diagonal. For example, the walk corresponding to $n = 6$ and $k = 0$



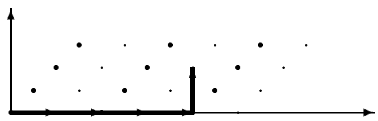
is transformed into the walk



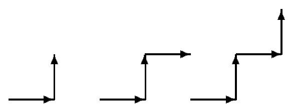
which corresponds to $n = 6$ and $k = 4$. The latter walk is obtained from the previous one by considering the symmetric piece of the walk (with respect to the even diagonal at distance two) contained between point $(2,0)$ and point $(3,3)$. If n is odd, we only need to change *even* with *odd*. For example, the walk corresponding to $n = 5$ and $k = 1$



is trasformed into the walk



and this corresponds to $n = 5$ and $k = 3$. The *stair walks* are not involved if we apply the previous transformation for the increasing values of k, since they are always situated between the first and second diagonals. In fact, if n is even, we have $k = 0$ and are not able to find any point of the walk on the even diagonal at distance two. The same is true when n is odd. Finally, if we give weight $+1$ and -1 to the $p_{n,k}^{[a,0]}$, we can conclude our proof because the number of *stair walks* is exactly a^n. For example, if $A = (1,0,1)$ and $n = 2,3,4$, the uninvolved stair walks are the following:



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Let us now examine another identity involving Shapiro's symmetric underdiagonal walks. To this purpose, we expand the Z^2 lattice so that every east and north step can have a general length, say z. We can prove the following theorem:

Theorem 4.5 *(Shapiro's identity) If $b' = b - a(z-1)$ we have:*

$$\sum_{k=0}^{n} s_{n,k}^{[a,b,b']}(zk+1) = (b+2a)^n.$$

Proof. The algebraic proof can be obtained by applying Theorem 2.1, if we consider that the generating function of the sequence $\{zk+1\}_{k\in N}$ is $f(t) = (1+(z-1)t)/((1-t)^2)$. As far as the combinatorial proof of Shapiro's identity is concerned, let us consider a line λ parallel to the main diagonal and at distance 1 from it. We define the weight of a walk as the distance of its ending point from λ; clearly, all the walks counted by $s_{n,k}^{[a,b,b']}$ have a weight of $zk+1$. The total weight of all of Shapiro's symmetric underdiagonal walks is therefore:

$$W_n = \sum_{k=0}^{n} s_{n,k}^{[a,b,b']}(zk+1),$$

and the theorem identity is proved if we prove that $W_n = (b+2a)^n$. Let us proceed by induction on n. The initial cases $W_0 = 1$ and $W_1 = b+2a$ are trivial and can be checked by inspection. Now, let us assume $W_n = (b+2a)^n$ and take a walk not ending on the main diagonal. Let w be its weight; the walk generates the following walks of length $n+1$: a walks of weight $(w+z)$, b walks of weight w and a walks of weight $(w-z)$. Clearly, the total balance for these walks is $(b+2a)w$. When we have a walk ending on the main diagonal, its weight is $w=1$ and it can only produce b' walks of weight w and a walks of weight $(w+z)$. The total balance is again $b'w + a(w+z) = (b+2a)w$ because $b' = b - a(z-1)$. Hence, every walk of length n and weight w produces some walks of length $n+1$ and a total weight of $(b+2a)w$. Since all the walks are obtained in this way, we conclude that $W_{n+1} = (b+2a)W_n = (b+2a)^{n+1}$. ∎

The following identities will only be proved algebraically. The first one is represented by a relation in which Fibonacci's number sequence F_n appear.

Theorem 4.6 *If* $\{f_k\} = \{0,1,1,0,0,0,-1,-1,0,0,0,1,1,0,0,0,-1,-1,\ldots\}$, *then*

$$\sum_{k=0}^{n} p_{n,k}^{[a,0]} f_k = F_n a^n.$$

Proof. The generating function of the sequence f_k is $f(t) = t/(1-t+t^2-t^3+t^4)$, and so for Theorem 2.1 the proof is complete. ∎

The last two identities concern the symmetric underdiagonal walks corresponding to values of a, b, c such that $b^2 - 4ac = 0$, and for this reason they represent an application of Theorem 3.6; in particular they correspond to the sequence $A = (1,2,1)$. These results are probably not very significant from a combinatorial point of view, but they evidence the validity of an algebraic approach in the search of closed forms for combinatorial sums.

Theorem 4.7 *If* $\{f_k\} = \{1,-1,0,1,-1,0,1,-1,0,1,-1\ldots\}$ *and* $\{g_k\} = \{0,1,-2,1,2,-4,2,3,-6,3,4,-8,4\ldots\}$, *then*

$$\sum_{k=0}^{n} p_{n,k}^{[1,2]} f_k = 1, \qquad \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k-1}{2k+1} = f_n$$

and

$$\sum_{k=0}^{n} p_{n,k}^{[1,2]} g_k = n, \qquad \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k-1}{2k+1} k = g_n.$$

Proof. The first identity derives from Theorem 2.1, with $f(t) = (t^2+t+1)^{-1}$. The second identity derives from Theorem 3.6, for which we have:

$$\bar{p}_{n,k}^{[1,2]} = (-1)^{n-k} \binom{n+k-1}{2k+1}.$$

The other identities are proved in an analogous way, if we consider that the generating function of the sequence g_k is $g(t) = t/(t^2+t+1)^2$. ∎

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