Fountains and histograms

Donatella Merlini, Renzo Sprugnoli
Dipartimento di Sistemi e Informatica
via Lombroso 6/17, 50134, Firenze, Italia

May, 2002

Abstract

We consider the well-known problem of how to arrange coins in a fountain and generalize it, by using combinatorial objects known as staircase polyominoes or p-histograms. We show how it is possible to obtain an asymptotic formula approximating the number of p-histograms of area n, as $n \to \infty$.

1 Introduction

An (n, k)-fountain is an arrangement of n coins in rows such that there are exactly k coins in the bottom row, and such that each coin in a higher row touches exactly two coins in the next lower row. Let $\varphi_{n,k}$ be the number of (n, k)-fountains and let $f_n = \sum_k \varphi_{n,k}$ be the number of fountains of n coins. The question of determining $\{f_n\}_{n\in\mathbf{N}}$ was raised by R. Guy in [6], who attributed it to J.G. Propp, and solved by Odlyzko and Wilf [9] who give the asymptotic formula $f_n \sim cx_0^{-n}$, where $c = 0.312363324596741\ldots$ and $x_0 = 0.576148769142756\ldots$ (see also below).

The (n, k)-fountains are in a 1-1 correspondence with Dyck paths and combinatorial objects known as "staircase polyominoes" (see, e.g., [4]). These are columns of squares such that the $(j + 1)$th column is at most one square higher than the jth column, and the first column just contains a single square. In Figure 1 we give a (14,8)-fountain and the corresponding Dyck path and staircase polyomino. It is clear that an (n, k)-fountain corresponds to a Dyck path of length $2k$ and area n and to a staircase polyomino having area n and k columns. Recently, staircase polyominoes (read from right to left) have been studied because they can represent the queue of a slow device attached to a computer, such as a printer (see, e.g., [7, 8]): every column is a time interval, for example the time necessary to print a page, so that a column is the number of users queuing at a certain moment. In a single interval many users can be added to the queue, but only one can be finished to be served. When the available printers are more than one, the queue can decrease by one or more users, and this requires a new model. Since the term "staircase polyomino" is not particularly appealing, a new terminology has been introduced: a *p-histogram* is a sequence of columns in which the height of the $(j + 1)$th column is at most $k + p$, if k is the height of column j. The first column has height r where $1 \leq r \leq p$ and the p-histograms with

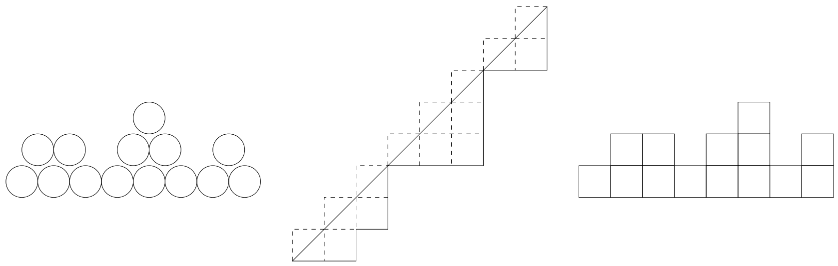

Figure 1: A fountain and the corresponding Dyck path and staircase polyomino.

the first column of height r will be called $p_{(r)}$-*histograms*. In this respect, p-histograms are the union of $p_{(r)}$-histograms (where $1 \leq r \leq p$) and staircase polyominoes correspond to $1_{(1)}$-histograms or 1-histograms. For $p > 1$ the correspondence with fountains is lost.

$p_{(1)}$-histograms are easily counted by number of columns and are related to generalized Dyck paths, i.e., paths on the $\mathbf{Z}^2$ lattice, starting at the origin, only composed by east and north steps and never going above the line $y = px$ (p a positive integer). These paths have length $(p+1)k$ and area n if properly counted (see, e.g., Figure 2). Their enumeration problems have been widely studied and recently Ph. Duchon [5] has given an explicit formula for the generating function enumerating these paths by length and area, but the problem remains of how to deduce an expression only depending on the area and so how to find an exact or asymptotic formula for the number of $p_{(1)}$-histograms having a given area. Independently, D. Merlini [7] found an analogous bivariate function for the whole set of p-histograms. The correspondence between $p_{(r)}$-histograms ($r > 1$) and generalized Dyck paths becomes artificial while $p_{(r)}$-histograms are related to $(p+1)$-ary trees as will be shown in Section 4.

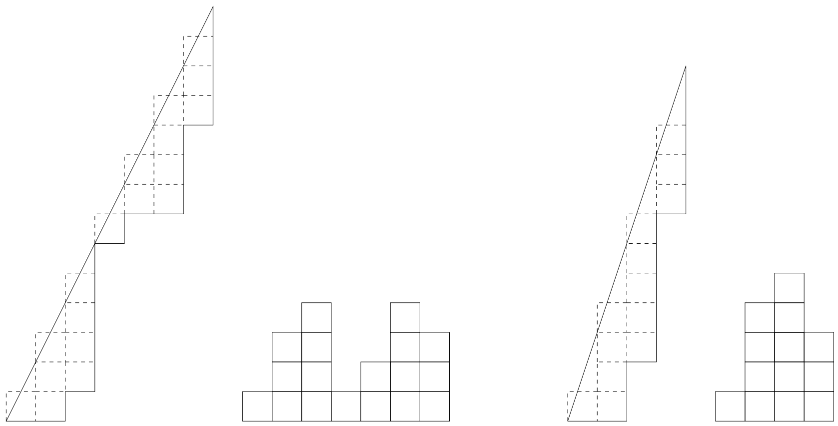

Figure 2: A $2_{(1)}$-histogram, a $3_{(1)}$-histogram and the corresponding generalized Dyck paths.

The study of p-histograms by area can be seen as a generalization of the coins in a fountain problem studied by Odlyzko and Wilf [9], which corresponds to the case $p = r = 1$. In that case, the authors find the generating function $F^{[1]}(z)$ counting the number of $p_{(1)}$-histograms by area and find in particular a relation with Ramanujan continued fraction:

$$F^{[1]}(z) = F(z) = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{z^2}{1 - \cfrac{z^3}{1 - \cdots}}}}$$

from which the asymptotic formula for $f_n^{[1]} = f_n$ can be deduced. For $p > 1$ things are a bit more complex and we approach the problem by using some combinatorial properties of p-histograms. Actually, in Section 2, we solve the case $p = 1$ obtaining again the Odlyzko and Wilf's result. In Section 3 we deal with the general case and finally, in Section 4, we study the problem from a different combinatorial point of view, relate p-histograms with other combinatorial objects and find an asymptotic expression for the radius of convergence of the corresponding generating functions. The enumeration of p-histograms by area gives rise, for $p > 1$, to sequences which do not appear in the Online Encyclopedia of Integer Sequences maintained by N. Sloane.

2 Histograms

In order to introduce the concept of p-histograms let us give some definitions: a *cell* is a 1×1 square and a *column* is a finite number of cells arranged one on top of the other. If we juxtapose several columns in such a way that their lowest cells are at the same level, we obtain what we call a *histogram*.

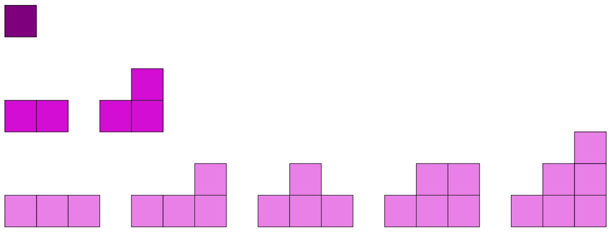

Figure 3: The $1^{[1]}$-histograms with at most 3 columns.

Definition 2.1 ($p_{(r)}$-histograms and their area) *For $p \geq 1$ and $r = 1 \ldots p$, a $p_{(r)}$-histogram of length k is a histogram whose first column only contains r cells and, whenever a column is composed by j cells, then the next column contains at most $j + p$ cells. Formally, a $p_{(r)}$-histogram of length k can be seen as a sequence of integers $H = (h_1, h_2, \ldots, h_k)$ such that $h_1 = r$ and $\forall i \in [2..k]$ $h_{i+1} \leq h_i + p$; the sum $\sum_{i=1}^{k} h_i$ represents the area of H.*

Figures 3 and 4 illustrate some examples of $p_{(r)}$-histograms.

Definition 2.2 (p-histograms) *For $p \geq 1$ the set of p-histograms of length k is the union of the sets of the $p_{(r)}$-histograms of length k with $r = 1 \ldots p$. Histograms correspond to p-histograms with $p = \infty$.*

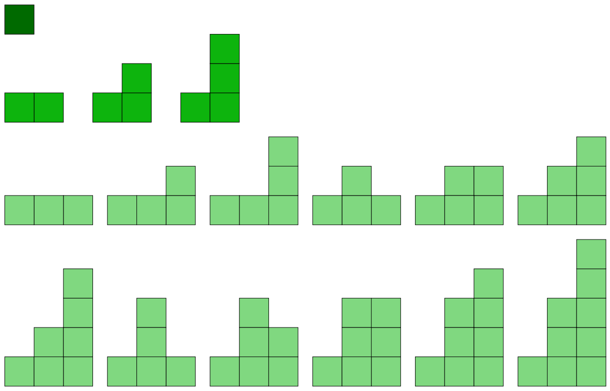

Figure 4: The $2^{[1]}$-histograms with at most 3 columns.

While the number of p-histograms with a given number of columns is finite, there exists an infinite number of histograms even with a single column. However, if we count them by area, we immediately get:

Lemma 2.3 *Let $f_n^{[\infty]}$ be the number of histograms of area n; then $f_n^{[\infty]} = 2^{n-1}$,* $\quad n > 0$.

Proof: For $n = 1$ we only have the histogram composed by a single cell; for $n = 2$ we have two histograms: the horizontal and vertical dominoes. For $n > 2$ we have a single column histogram composed by n cells or the first column can contain $j = 1, 2, \ldots, n-1$ squares and is followed by a histogram of area $n - j$. Therefore we have:

$$f_n^{[\infty]} = f_{n-1}^{[\infty]} + f_{n-2}^{[\infty]} + \cdots + f_1^{[\infty]} + 1$$

and this recurrence is easily solved. ■

Let $f_n^{[p]}$ be the number of p-histograms of area n and set $F^{[p]}(z) = \sum_{n=0}^{\infty} f_n^{[p]} t^n$; since $f_n^{[p]} \leq f_n^{[\infty]}$, $\forall n \in \mathbf{N}$, we can state the following:

Corollary 2.4 *The radius of convergence of the series $F^{[p]}(z)$ is $\geq 1/2$.*

We can somewhat strengthen this result by the following construction. Let us consider a p-histogram H of area n; we can obtain from it one or two new p-histograms of area $n+1$ in the following way:

- add to H a new last column containing a single square;
- add to H a new square on top of its last column, provided that in this way the last column remains of height $\leq j + p$, j being the height of the last-but-one column.

It is immediate to see that all the p-histograms of area $n+1$ are obtained, once and only once, by this construction from the p-histograms of area n; therefore, as a matter of fact, we can build all the p-histograms by starting with the empty one and recursively applying the construction. For example, in Figure 5 we show the first steps in the construction of 1-histograms. We obtain:

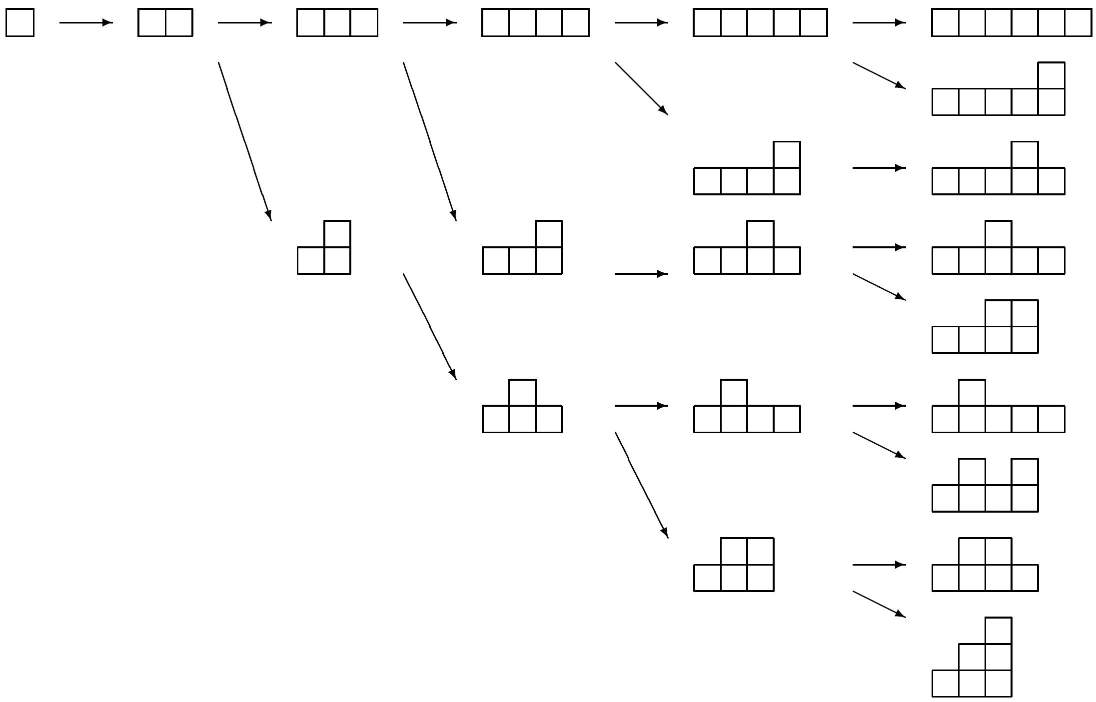

Figure 5: The generation of 1-histograms

Lemma 2.5 *The radius of convergence ρ of $F^{[p]}(z)$ is $1/2 < \rho < \phi$, where ϕ is the golden ratio $\phi = (\sqrt{5}-1)/2 \approx 0.618034$.*

Proof: Let us suppose we have obtained the $f_n^{[p]}$ p-histograms by the previous construction; if we go on another step, we obtain $f_{n+1}^{[p]}$ p-histograms, $f_n^{[p]}$ of which are terminated by a column with a single square. By performing another step, we obtain $f_{n+2}^{[p]}$ p-histograms of which:

- $f_{n+1}^{[p]}$ have a terminating one-square column;
- $f_n^{[p]}$ have a terminating two-square column (they are obtained from the p-histograms of area $n+1$ having a terminating one-square column).

Therefore we have $f_{n+2}^{[p]} \geq f_{n+1}^{[p]} + f_n^{[p]}$; by comparing this formula with the recurrence for Fibonacci numbers we immediately infer the upper bound ϕ for the radius of convergence. ■

Let us denote by $f_{m,j}^{[p]}$ the number of p histograms of area m having their last column of height j. If we delete this last column, we obtain a p histogram of area $m-j$ having a last column of height $j-p$, or higher. This means:

$$f_{m,j}^{[p]} = f_{m-j,j-p}^{[p]} + f_{m-j,j-p+1}^{[p]} + f_{m-j,j-p+2}^{[p]} + \cdots . \tag{2.1}$$

$n\backslash k$	1	2	3	4	1	2	3	4	5	6	1	2	3	4	5	6	7
1	1				1						1						
2	1				1	1					1	1					
3	1	1			2	1					2	1	1				
4	2	1			3	2	1				4	2	1				
5	3	2			6	3	2				7	4	2	1			
6	5	3	1		11	6	3	1			14	7	4	2			
7	9	5	1		21	11	6	1			27	14	7	4	1		
8	15	9	2		39	21	11	3			53	27	14	7	2		
9	26	15	4		74	39	21	5	1		103	53	27	14	3	1	
10	45	26	6	1	140	74	39	10	2		201	103	53	27	7	1	
11	78	45	11	1	265	140	74	18	4		392	201	103	53	13	3	
12	135	78	19	2	501	265	140	35	7	1	765	392	201	103	26	6	1

Table 2.1: The 1- 2- and 3-histograms counted by height of their last column

Figure 5 shows the construction of 1-histograms up to area 6; by grouping these histograms according to the height of their last column, the values in the upper part of the first triangle in Table 2.1 is easily checked.

Let us denote by $f_k(z) = \sum_{n=0}^{\infty} f_{n,k}^{[1]} z^n = \sum_{n=0}^{\infty} f_{n,k} z^n$ the generating function of the k^{th} column in the infinite triangle corresponding to the case $p = 1$.

From (2.1) we have:

$$f_{n+1,1} = f_{n,0} + f_{n,1} + f_{n,2} + f_{n,3} + \cdots$$

and, by passing to generating functions:

$$\frac{f_1(z)}{z} = f_0(z) + f_1(z) + f_2(z) + \cdots = 1 + f_1(z) + f_2(z) + \cdots .$$

Analogously, for the elements in column 2 we have:

$$f_{n+1,2} = f_{n-1,1} + f_{n-1,2} + f_{n-1,3} + \cdots \qquad \text{or} \qquad f_{n+2,2} = f_{n,1} + f_{n,2} + f_{n,3} + \cdots .$$

Again, for the generating functions we find:

$$\frac{f_2(z)}{z^2} = f_1(z) + f_2(z) + f_3(z) + \cdots .$$

If we put together these last two relations, we have:

$$\frac{f_1(z)}{z} = 1 + \frac{f_2(z)}{z^2} \qquad \text{or} \qquad f_2(z) = zf_1(z) - z^2.$$

What we are looking for is the generating function for the row sums of the triangle, which, by construction, is just $f_1(z)/z$; so, let us call $F(z)$ this unknown function; we immediately have:

$$f_1(z) = zF(z) \qquad \text{and} \qquad f_2(z) = z^2F(z) - z^3.$$

We can now proceed as before by considering column 3:

$$f_{n+3,3} = f_{n,2} + f_{n,3} + f_{n,4} + \cdots \qquad \text{or} \qquad \frac{f_3(z)}{z^3} = f_2(z) + f_3(z) + f_4(z) + \cdots .$$

Now we have:

$$\frac{f_3(z)}{z^3} = \frac{f_2(z)}{z^2} - f_1(z) \qquad \text{or} \qquad f_3(z) = zf_2(z) - z^3 f_1(z).$$

We could compute an explicit formula for $f_3(z)$ in terms of $F(z)$, but it is a simpler matter to observe that, in general, we have:

$$f_s(z) = zf_{s-1}(z) - z^s f_{s-2}(z),$$

from which a complete recurrence is found, when we add $f_1(z)$ and $f_2(z)$ as initial values. In principle, we can compute $f_s(z)$ for every $s \in \mathbf{N}$, but it is not strictly necessary. Let us begin with the following:

Lemma 2.6 *We have* $\mathrm{ord}(f_s(z)) = s(s+1)/2$.

Proof: There exists a single histogram of minimal area and with its last column of height s : the histogram with columns of heights $1, 2, \ldots, s$. Its area is therefore $1 + 2 + \cdots + s = s(s+1)/2$. ■

Let us now observe that:

$$F(z) = 1 + f_1(z) + f_2(z) + f_3(z) + \cdots ;$$

if we stop the sum at $f_s(z)$, the first $s(s+1)/2$ terms of $F_s(z) = 1 + f_1(z) + f_2(z) + \cdots + f_s(z)$ are equal to the coefficients of $F(z)$ and their radii of convergence tends to be the same. So, by the previous recurrence, we can express $f_1(z)$, $f_2(z)$, ..., $f_s(z)$ in terms of $F(z)$ and solve the truncated identity in $F(z) \approx F_s(z)$. The function $F_s(z)$ is a rational approximation of $F(z)$ and by using Maple we find for $s = 12$ and $s = 13$ (Digits=20):

$$\rho \approx 0.57614876914275$$

as the minimal positive real root of the denominator in $F_s(z)$; this root, by lemma 2.5, belongs to the interval $[1/2, \phi]$. The simplicity of this root will be proved in Section 4 where we'll find the following approximation (see also [9]):

$$f_n^{[1]} \sim C\rho^{-n}.$$

The constant C can be easily determined as:

$$\lim_{s\to\infty} \left(\lim_{z\to\rho} (1 - z/\rho) F_s(z) \right) = C;$$

for $s = 12$ and $s = 13$ we find the same value:

$$C \approx 0.3123633245967414$$

which is also the value found by Odlyzko and Wilf.

3 The general case

The method applied in the previous section can now be extended to the general case $p \geq 1$. We begin by adding a row and a column 0 to the infinite triangles $\{f^{[p]}_{n,j}\}_{n,j \in N}$ counting p-histograms of area n by the height k of their last column; the added elements are all 0, except $f^{[p]}_{0,0} = 1$, corresponding to the empty p-histogram, the only one having a "last" column of height 0. Now, by using equation (2.1) we can obtain a recurrence relation for the column generating functions of our triangles. Let us begin by the following:

Lemma 3.1 *If $F^{[p]}(z)$ is the generating function for the row sums of the triangle of p-histograms, i.e., is the generating function counting p-histograms by area, and $f^{[p]}_j(z)$ is the generating function of the jth column, then we have:*

$$\begin{aligned} f^{[p]}_0(z) &= 1 \\ f^{[p]}_j(z) &= z^j F^{[p]}(z) \quad \text{for} \quad 0 < j \leq p. \end{aligned}$$

Proof: Column 0 has 1 as its generating function, as we have already observed. By equation (2.1), $f^{[p]}_{n,j}$, $(0 < j \leq p)$, is just the sum of the row located j positions above n, i.e., $F^{[p]}(z)$ translated by j positions. This completes our proof. ■

This lemma is the first step to show that the order of the column generating functions grows sufficiently fast to make effective our approach, in the sense that will become clear as we proceed. Formally, we have a result generalizing Lemma 2.6:

Theorem 3.2 *Let $f^{[p]}_j(z)$ be the generating function of column j in our infinite triangles; then* $\operatorname{ord}(f^{[p]}_j(z)) = \Theta(j^2)$.

Proof: We proceed by computing the minimum area of a p-histogram with a column of height j; this is given by the following sum:

$$\sum_{k=0}^{\lceil \frac{j}{p} \rceil} (j - kp) = j \lceil \frac{j}{p} \rceil - \frac{p}{2} \lceil \frac{j}{p} \rceil \left(\lceil \frac{j}{p} \rceil - 1 \right) = \frac{j^2}{2p} + \frac{j}{p} + \frac{p\delta}{2} - \frac{p\delta^2}{2}$$

where we used $\lceil \frac{j}{p} \rceil = \frac{j}{p} + \delta$ and $0 \leq \delta < 1$. This proves the theorem. ■

This result is important in our approach. In fact, we intend to apply a procedure analogous to the one used in the previous section, and it is effective only if the quantity of computations to arrive to a sufficient approximation of $\rho^{[p]}$, the radius of convergence for $F^{[p]}(z)$, is limited. This quantity is directly related to the number of terms for which $F^{[p]}(z)$ coincides with $F^{[p]}_s(z) = \sum_{j=1}^{s} f^{[p]}_j(z)$, which increases as the order of $f^{[p]}_s(z)$. The quadratic growth of this order is a guarantee that our procedure yields rapidly a good approximation for $\rho^{[p]}$.

Now we are ready for our recurrence relation:

Theorem 3.3 *The column generating functions for the triangles $\{f^{[p]}_{n,k}\}_{n,k\in N}$ counting p-histograms of area n by height k of their last column are related by the following recurrence:*

$$f^{[p]}_j(z) = zf^{[p]}_{j-1}(z) - z^j f^{[p]}_{j-p-1}(z).$$

Proof: The equation (2.1) can be written as:

$$f^{[p]}_{n+j,j} = f^{[p]}_{n,j-p} + f^{[p]}_{n,j-p+1} + f^{[p]}_{n,j-p+2} + \cdots ,$$

which is equivalent, in terms of column generating functions, to:

$$\frac{f^{[p]}_j(z) - c_j(z)}{z^j} = f^{[p]}_{j-p}(z) + f^{[p]}_{j-p+1}(z) + f^{[p]}_{j-p+2}(z) + \cdots ,$$

$c_j(z)$ being composed by the first j terms of $f^{[p]}_j(z)$. The same relation for $j-1$ is:

$$\frac{f^{[p]}_{j-1}(z) - c_{j-1}(z)}{z^{j-1}} = f^{[p]}_{j-p-1}(z) + f^{[p]}_{j-p}(z) + f^{[p]}_{j-p+1}(z) + \cdots .$$

By Lemma 3.1 and Theorem 3.2 we have $c_j(z) = c_{j-1}(z) = 0$ and therefore we find:

$$\frac{f^{[p]}_j(z)}{z^j} = \frac{f^{[p]}_{j-1}(z)}{z^{j-1}} - f^{[p]}_{j-p-1}(z)$$

which is just the recurrence relation we were looking for. ■

At this point, we can easily devise a Maple procedure computing both the radius of convergence $\rho^{[p]}$ of $F^{[p]}(z)$ and the coefficient $C^{[p]}$, under the assumption that $\rho^{[p]}$ is a simple pole, that is $f^{[p]}_n \approx C^{[p]}(\rho^{[p]})^{-n}$. In Section 4 we'll show that this is in fact the case and prove that both the radius of convergence $\rho^{[p]}$ and the constant $C^{[p]}$ tend to $1/2$ as $p \to \infty$. In Table 3.2 we give some sample values obtained with our procedure. As we can see from this table, the values of $\rho^{[p]}$ are rapidly approaching 0.5, while the values of $C^{[p]}$ grow up to $p = 6$ and then decrease to 0.5. In the next section we develop a different approach, directly related to the bivariate generating function of Merlini and Duchon thus finding more precise information on the asymptotic behaviour of $\rho^{[p]}$ and $C^{[p]}$.

4 Another approach

Let us now consider a different approach based on the concept of generating trees. This concept has been used from various points of view and has been introduced in the literature by Chung, Graham, Hoggat and Kleiman [2] to examine the reduced Baxter permutations. This technique has been successively applied to other classes of permutations and more recently to various classes of combinatorial objects (see, e.g., [1, 10, 11]). Our approach remembers the methodology formalized in [1].

Definition 4.1 (Generating trees) *A generating tree is a rooted labeled tree with the property that if v_1 and v_2 are any two nodes with the same label then, for each label l, v_1 and v_2 have exactly the same number of children with label l. To specify a generating tree it therefore suffices to specify:*

p	ρ	C
1	0.57614876914275	0.31236332459675
2	0.52809811886833	0.44616470348035
3	0.51220594097366	0.48698755866208
4	0.50567621390479	0.49916037281165
5	0.50272965282290	0.50217160982755
6	0.50133606906489	0.50234398759604
7	0.50066026591343	0.50179909956393
8	0.50032802044520	0.50121505953625
9	0.50016343551669	0.50076657067668
10	0.50008156181213	0.50046337737673

Table 3.2: Some sample values

1) the label of the root;

2) a set of rules explaining how to derive from the label of a parent the labels of all of its children.

For example, Figure 6 illustrates the upper part of the generating tree which corresponds to the following specification:

$$\begin{cases} root: & (1) \\ rule: & (k) \quad \rightarrow (1)\cdots(k)(k+1) \end{cases} \tag{4.2}$$

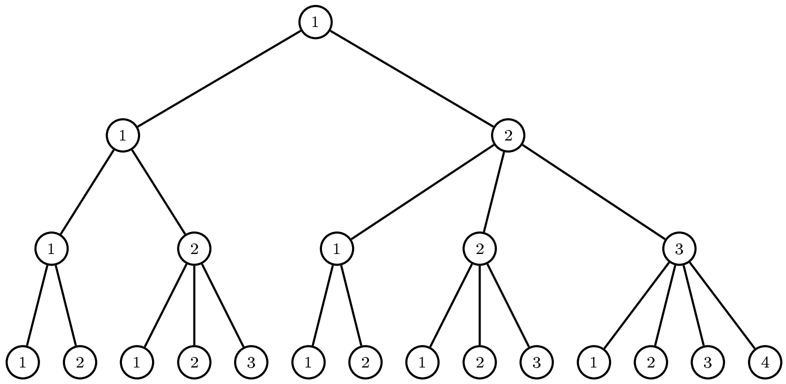

Figure 6: The partial generating tree for the specifications (4.2) and (4.3) when $p = 1$.

It is at all obvious that a $p_{(r)}$-histogram corresponds to a path in the generating tree produced by the following specification:

$$\begin{cases} root: & (r) \\ rule: & (k) \quad \rightarrow (1)\cdots(k+p-1)(k+p), \qquad p \geq 1 \end{cases} \tag{4.3}$$

which corresponds to the specification (4.2) when $p = r = 1$. In fact, it is sufficient to associate the sequence of column heights of the histogram (from left to right) to the labels in a particular path in the tree. Figure 3 illustrates the $1^{[1]}$-histograms with at most 3 columns which correspond to the generating tree in Figure 6; Figures 7 and 4 illustrate the case $p = 2$ and $r = 1$.

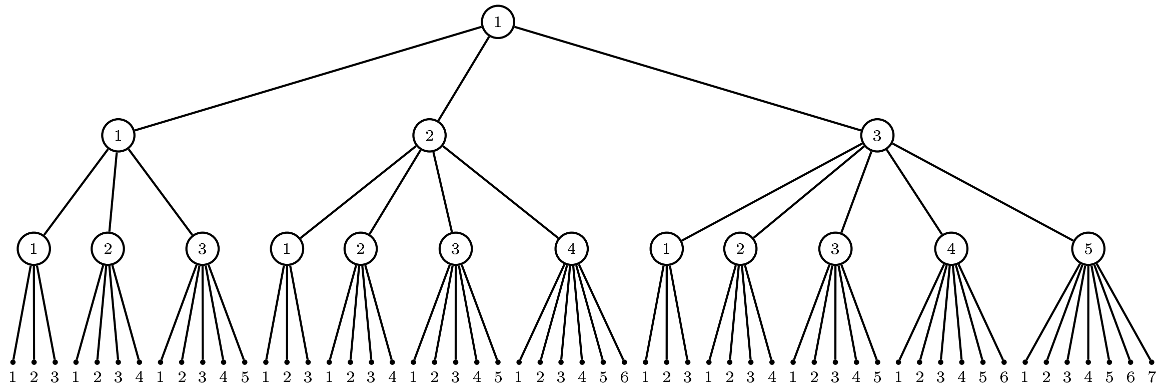

Figure 7: The partial generating tree for the specification (4.3) when $p = 2$ and $r = 1$.

We now show a bijection between the generating tree specification (4.3) with $r = 1$ and $(p+1)$-ary trees which will be very useful in our development; to this purpose, we introduce a labelling technique by giving the definition of *e-labeled* $(p+1)$*-ary trees* (the e comes from the term *embedding* below):

Definition 4.2 *An e-labeled* $(p+1)$*-ary tree is a* $(p+1)$*-ary tree whose nodes are labeled in the following way:*

1) if the $(p+1)$*-ary tree is complete (that is, no subtrees are empty) then we label the nodes in the following way:*

- *we label the root by* 1;
- *recursively, if a node is labeled* j*, then we label its children by* $j, j+1, \ldots, j+p$ *proceeding from right to left.*

2) if the $(p+1)$*-ary tree is not complete we label its nodes by embedding it into a complete* $(p+1)$*-ary tree labeled as in 1).*

Figure 8 illustrates a complete e-labeled ternary tree. The relation between these trees and generating trees is given by the following theorem.

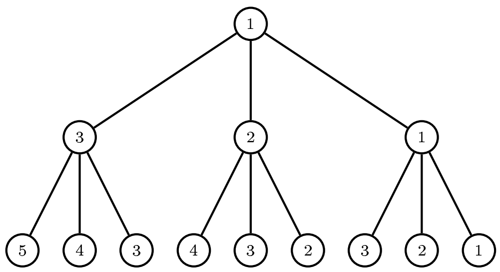

Figure 8: A complete e-labeled ternary tree.

Theorem 4.3 *There exists a $1-1$ correspondence between the set of e-labeled $(p+1)$-ary trees with $k+1$ nodes and the nodes at level k in the generating tree corresponding to the specification (4.3) with $r=1$ (0 being the label of the root).*

Proof: Every node at level k in the generating tree determines a unique $(k+1)$-path $(h_0, h_1, \ldots, h_k)$ in the generating tree; therefore, what we will show is that there is a $1-1$ correspondence between e-labeled $(p+1)$-ary trees and paths in the generating tree. Let us define the procedure to go from a $(k+1)$-path $(h_0, h_1, \ldots, h_k)$ to the corresponding e-labeled $(p+1)$-ary tree:

- the first element h_0 in the $(k+1)$-path is always 1; let it correspond to the root of the tree, labeled 1. Connect the root by means of dotted lines to all its children, labeled 1 through $p+1$ from right to left;
- after proceeding as far in the path as an element with value h_{i-1}, we will have constructed an e-labeled tree in which some of the edges have only tentative status (denoted by dotted lines instead of solid ones). Now, if h_i is the next element in the path we should have $1 \leq h_i \leq h_{i-1}+p$: then we change the dotted line arriving to h_i (not necessarily coming from h_{i-1}) into a solid line, delete all the dotted lines on its left and finally connect h_i by means of dotted lines to all its children, labeled h_i through h_i+p from right to left.
- as a final step, when the $(k+1)$-path is exhausted, we eliminate all the dotted lines and the nodes at their ends.

The inverse correspondence is now rather obvious; suppose we have an e-labeled $(p+1)$-ary tree with $k+1$ nodes and simply perform a pre-order visit to the tree. What we get is a $(k+1)$-path in the generating tree because, by construction, its elements satisfy the condition for $(k+1)$-paths, i.e., an element h_{i-1} can only be followed by a h_i such that $1 \leq h_i \leq h_{i-1}+p$. Obviously, different $(k+1)$-paths correspond to different e-labeled trees, and the proof is complete. ∎

We are now in a position to relate histograms, trees and generating trees:

Theorem 4.4 *There exists a $1-1$ correspondence between the $p_{(1)}$-histograms with k columns and the k-length paths in the generating tree produced by the specification (4.3) with $r=1$ and hence with $(p+1)$-ary trees with k nodes. In particular, each label in the $(p+1)$-ary tree equals the height of the corresponding column in the associated $p_{(1)}$-histogram.*

Proof: The first correspondence follows immediately from the initial considerations; the correspondence with $(p+1)$-ary trees comes from Theorem 4.3. ∎

An obvious consequence of this fact is that the $p_{(1)}$-histograms with k-columns are counted by the generating function $T(t)$ counting the number T_k of $(p+1)$-ary trees with k nodes:

$$T(t) = 1 + tT(t)^{p+1} \tag{4.4}$$

if we count as 1 the empty histogram, i.e., the histogram containing no cell.

As it is well known, except for $p=1$, when we have $T(t)=(1-\sqrt{1-4t})/2t$, an explicit formula for $T(t)$ is awkward or is impossible to obtain (for $p \geq 4$). However, an expression for T_k, is easily found by means of the Lagrange Inversion Formula. By setting $T_k=[t^k]T(t)$ and $w=w(t)=T(t)-1$ we obtain:

$$T_k = [t^k]\left[w+1 \middle| w = t(w+1)^{(p+1)}\right] = \frac{1}{k}[t^{k-1}](t+1)^{(p+1)k} =$$

$$= \frac{1}{k}\binom{(p+1)k}{k-1} = \frac{1}{pk+1}\binom{(p+1)k}{k};$$

this is the famous formula counting $(p+1)$-ary trees; obviously, when $p=1$, it gives an explicit expression for the Catalan numbers.

Let us denote by $A(t,z)$ the bivariate generating function counting the number $[t^k][z^n]A(t,z)$ of $p_{(1)}$-histograms of length k and area n. We have the following result:

Theorem 4.5 *The bivariate generating function $A(t,z)$ satisfies the following functional equation:*

$$A(t,z) = 1 + tzA(t,z)A(tz,z)\dots A(tz^{p-1},z)A(tz^p,z). \tag{4.5}$$

Proof: The proof follows from Theorem 4.4. In fact $A(t,z)$ also counts the number of e-labeled $(p+1)$-ary tree with k nodes and in which the sum of the labels is equal to n. We simply note that $A(tz^i,t)$ refers to the number of e-labeled $(p+1)$-ary trees in which the root is labeled by $i+1$ (and, consequently, every label's node is increased by i with respect to Definition 4.2). Finally, the term 1 in the functional equation counts the empty tree. ■

Up to now, we found a series of results concerning $p_{(1)}$-histograms: these results will now be used to determine analogous results for p-histograms. We begin by giving the following definition:

Definition 4.6 *A $(p,p+1)$-ary tree is a tree composed by a root which has p children (possibly empty) each of which corresponds to a $(p+1)$-ary tree.*

We can label a $(p,p+1)$-ary tree by using a technique similar to the one used in Definition 4.2 (we call *e-labeled $(p,p+1)$-ary tree* the corresponding tree); in particular, we increase by 1 the labels in the $(p+1)$-ary trees as we proceed from right to left in the labeling of the p children of the root (see Figure 9).

Theorem 4.7 *There exists a $1-1$ correspondence between the set of e-labeled $(p,p+1)$-ary trees with k nodes (root excluded) and p-histograms with k columns.*

Proof: The proof parallels the one of Theorem 4.3 and we leave the details to the interested reader. ■

Let us denote by $\mathcal{A}(t,z)$ the generating function counting the number $[t^n][z^k]\mathcal{A}(t,z)$ of p-histograms of length k and area n. We have the following result:

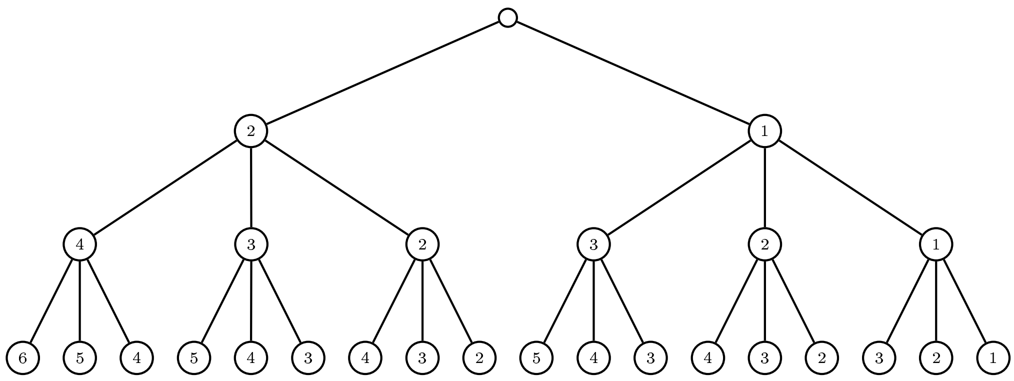

Figure 9: A complete e-labeled $(2, 3)$-tree.

Theorem 4.8 *The bivariate generating function $\mathcal{A}(t, w)$ satisfies the following equation:*

$$\mathcal{A}(t, z) = A(t, z)A(tz, z) \dots A(tz^{p-1}, z) = \frac{A(t, z) - 1}{tzA(tz^p, z)}. \tag{4.6}$$

Proof: The proof follows from Theorem 4.7. In fact $\mathcal{A}(t, z)$ also counts the number of e-labeled $(p, p+1)$-ary tree with k nodes (root excluded) and in which the sum of the labels is equal to n. This implies the formula

$$\mathcal{A}(t, z) = A(t, z)A(tz, z) \dots A(tz^{p-1}, z).$$

By substituting this expression into formula 4.5 we immediately get formula 4.6. ■

Corollary 4.9 *The number $\mathcal{T}_k$ of $(p, p+1)$-ary trees with k nodes (root excluded) and hence of p-histograms of length k is given by the following formula:*

$$\mathcal{T}_k = [t^k]\frac{T(t)^p - 1}{t} = \frac{p}{k}\binom{p(k+1) + k - 1}{k - 1}$$

Proof: We simply observe that $\mathcal{T}(t) = \mathcal{A}(t, 1)$ and that $T(t) = A(t, 1)$. ■

The solution of equation (4.5) can be found by the following method, as proposed by Ph. Duchon [5]. Let us set $A(t, z) = G(tz, z)/G(t, z)$, so that (4.5) becomes:

$$G(tz, z) = G(t, z) + tzG(tz^{p+1}, z); \tag{4.7}$$

if we set: $G(t, z) = \sum_{k=0}^{\infty} a_k(z)t^k$ with $a_0(z) = 1$, by extracting the coefficient of t^k from the last equation, we obtain:

$$z^k a_k(z) = a_k(z) + a_{k-1}(z)z^{(k-1)(p+1)+1}.$$

By unfolding this recurrence, we arrive to:

$$a_k(z) = \frac{z^{k+k(k-1)(p+1)/2}}{(z^k - 1) \cdots (z - 1)},$$

that is (see Odlyzko and Wilf [9] for the case $p = 1$):

$$A(t,z) = \left(\sum_{k=0}^{\infty} \frac{z^{2k+k(k-1)(p+1)/2} t^k}{(z^k-1)\cdots(z-1)} \right) \Big/ \left(\sum_{k=0}^{\infty} \frac{z^{k+k(k-1)(p+1)/2} t^k}{(z^k-1)\cdots(z-1)} \right).$$

However, we are more interested in $\mathcal{A}(t,z)$, for which we have:

$$\mathcal{A}(t,z) = \frac{A(t,z)-1}{tzA(tz^p,z)} = \left(\frac{G(tz,z)}{G(t,z)} - 1 \right) \Big/ \left(tz\frac{G(tz^{p+1},z)}{G(tz^p,z)} \right) =$$

$$= tz\frac{G(tz^{p+1},z)}{G(t,z)} \cdot \frac{G(tz^p,z)}{tzG(tz^{p+1},z)} = \frac{G(tz^p,z)}{G(t,z)}.$$

Our original generating function $F^{[p]}(z) = \sum_{n=0}^{\infty} f_n^{[p]} z^n$ is now given by:

$$F^{[p]}(z) = \mathcal{A}(1,z) = \frac{G(z^p,z)}{G(1,z)};$$

$G(z^p,z)$ and $G(1,z)$ are analytic in $|z| < 1$ and therefore the radius of convergence $\rho^{[p]}$ of $F^{[p]}(z)$ is given by the zero of $G(1,z)$ of minimal modulus.

Let us begin by setting $t = 1$ in equation (4.7); it becomes:

$$G(1,z) = G(z,z) - zG(z^{p+1},z) \tag{4.8}$$

and this shows that $G(1,z)$ can be written in terms of various $G(z^m,z)$, for $m \geq 1$. In order to simplify notations, let us write Φ_m for $G(z^m,z)$, so that equation (4.8) is written: $\Phi_0 = \Phi_1 - z\Phi_{p+1}$. Besides, if we set $t = z^m$ in equation (4.7) we obtain:

$$\Phi_m = \Phi_{m+1} - z^{m+1}\Phi_{m+p+1} \tag{4.9}$$

and we can substitute, for $m = 1$, the resulting value for $G(z,z) = \Phi_1$ into (4.8):

$$\Phi_0 = \Phi_2 - w\Phi_{p+1} - z^2\Phi_{p+2}.$$

We can now continue in the same way and substitute the value for Φ_2, obtained from (4.9). It is therefore a simple matter to prove, by mathematical induction, that:

$$\Phi_0 = P_m^{[0]}(z)\Phi_m + P_m^{[1]}(z)\Phi_{m+1} + \cdots,$$

where $P_m^{[i]}(z), i \geq 0$ are polynomials in z. What we are now interested in is the polynomial $P_m^{[0]}(z)$ and the "order" of the other polynomials, that is the smallest degree of their monomials. The first p applications of equation (4.9) give the following relation:

$$\Phi_0 = \Phi_p - z\Phi_{p+1} - z^2\Phi_{p+2} - \cdots - z^p\Phi_{2p},$$

because every step simply adds a term to the right hand member. During the next p steps, the polynomials $P_m^{[i]}(z), i \geq 0$ begin to be formed in a simple way; the resulting equation is:

$$\Phi_0 = (1 - z - \cdots - z^p)\Phi_{2p} - z^{p+1}\Phi_{2p+1} - z^{p+2}(1-z)\Phi_{2p+2} -$$

$$-z^{p+3}(1-z-z^2)\Phi_{2p+3}\cdots - z^{2p}(1-z-\cdots-z^{p-1})\Phi_{3p}.$$

If we perform other p steps, the various polynomials are summed up to $\Phi_0 = P_{3p}^{[0]}(z)\Phi_{3p}+\cdots$ where

$$P_{3p}^{[0]}(z) = 1-z-\cdots-z^p-z^{p+1}-z^{p+2}(1-z)\cdots-z^{2p}(1-z-\cdots-z^{p-1}),$$

and, by using the identity:

$$1-z-z^2\cdots-z^{j-1} = 2-\frac{1-z^j}{1-z} = \frac{1-2z+z^j}{1-z} \qquad \text{for } j\geq 1,$$

we have:

$$P_{3p}^{[0]}(z) = \frac{1-2z+z^{p+1}}{1-z} - \sum_{j=1}^{p} z^{p+j}\frac{1-2z+z^j}{1-z}.$$

We can now extend the summation to infinity by introducing an error in the order of z^{3p}, i.e. $1/8^p$. After some computations we find:

$$P_{3p}^{[0]}(z) \sim \frac{1-2z+z^{p+1}}{1-z} - \frac{z^{p+1}-2z^{p+3}}{(1-z)(1-z^2)} = \frac{(1-2z)(1-z^2)+z^{p+3}}{(1-z)^2(1+z)}.$$

The last expression for $P_{3p}^{[0]}(z)$ shows the existence of the three zeroes $1/2, 1, -1$ when p tends to infinity and thus we have the proof that $\rho^{[p]}$ is simple and approaches $1/2$ as p increases, that is $f_n^{[p]} \sim C^{[p]}(\rho^{[p]})^{-n}$. At this point, we can find an approximation for $\rho^{[p]}$ by using the Newton method with initial value $z_0 = 1/2$ (see, e.g., [3, p.30]). We have:

$$(1-z)P_{3p}^{[0]}(z) = 1-2z+\frac{z^{p+3}}{1-z^2}$$

and from this equation the root $\rho^{[p]} \approx 1/2$ is apparent. The Newton method applied in a formal way gives:

$$\rho^{[p]} \sim \frac{1}{2} + \frac{1}{12\cdot 2^p} + \frac{3p+11}{216\cdot 4^p}.$$

For $p=5$ the approximated value is 0.5027217159 with a relative error of 0.0016% (see Table 3.2). For what concerns the constant $C^{[p]}$ we have:

$$C^{[\infty]} = \lim_{p\to\infty}\lim_{z\to\rho}(1-z/\rho)F^{[p]}(z) = \lim_{p\to\infty}\lim_{z\to\rho}(1-z/\rho)\frac{G(z^p,z)}{G(1,z)} = \frac{1}{2}.$$

In fact, $G(z^p,z)\to G(0,\rho) = a_0(\rho) = 1$, and $(1-2z)/G(1,z) \sim (1-2z)/P_{3p}^{[0]}(z) = 1-z \to 1/2$. We can thus conclude with the following theorem:

Theorem 4.10 *The number of p-histograms of area n can be approximated by the formula* $f_n^{[p]} \sim C^{[p]}(\rho^{[p]})^{-n}$ *with*

$$\rho^{[p]} \sim \frac{1}{2} + \frac{1}{12\cdot 2^p} + \frac{3p+11}{216\cdot 4^p}.$$

Both the radius of convergence $\rho^{[p]}$ *and the constant* $C^{[p]}$ *tend to* $1/2$ *as* $p\to\infty$.

The constant $C^{[p]}$ can be determined numerically as a function of $\rho^{[p]}$ as we did in the previous section.

Acknowledgements

We wish to thank the anonymous referees for the useful suggestions to our paper.

References

[1] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. Eco: a methodology for the Enumeration of Combinatorial Objects. *Journal of Difference Equations and Applications*, 5:435–490, 1999.

[2] F. R. K. Chung, R. L. Graham, V. E. Hoggat, and M. Kleiman. The number of Baxter permutations. *Journal of Combinatorial Theory, Series A*, 24:382–394, 1978.

[3] N. G. de Bruijn. *Asymptotic methods in analysis.* North-Holland Publishing Co., Amsterdam, 1958.

[4] M. P. Delest and S. Dulucq. Enumeration of directed column-convex animals with given perimeters and area. *Croat. Chem. Acta*, 66:59–80, 1993.

[5] Ph. Duchon. On the enumeration and generation of generalized Dyck words. *Discrete Mathematics*, 225:121–135, 2000.

[6] R. Guy. The strong law of small numbers. *American Mathematical Monthly*, 95:697–712, 1988.

[7] D. Merlini. A generating tree approach to schedule printing devices, The Sixth Seminar on the Analysis of Algorithms, Krynica Morska, Danzica, July 2000.

[8] D. Merlini, R. Sprugnoli, and M. C. Verri. Waiting patterns for a printer. In *FUN with algorithms 2, E. Lodi, L. Pagli, N. Santoro, Editors*, Carleton Scientific, pages 183–198, 2001.

[9] A. M. Odlyzko and H. S. Wilf. The editor's corner: n coins in a fountain. *American Mathematical Monthly*, 95:840–843, 1988.

[10] J. West. Generating trees and the Catalan and Schröder numbers. *Discrete Mathematics*, 146:247–262, 1995.

[11] J. West. Generating trees and forbidden subsequences. *Discrete Mathematics*, 157:363–374, 1996.