Proving a Class of Hypergeometric Identities by means of Generating Functions

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Abstract. We propose a method to solve a class of terminating hypergeometric series ${}_2F_1$ by using generating functions and the Lagrange inversion formula. Altough the approach is not as general as the WZ-method, nonetheless it is rather effective and succeeds in solving sums that cannot be treated by similar methods, as the ones proposed for the solutions of Gosper's strange identities.

1 Introduction

In the theory of hypergeometric series there are many general results involving their evaluation for specific argument values. Almost all classical theorems are concerned with evaluation at 1 or -1, but in recent years many efforts have been profused in evaluating series, mainly terminating series, in other points such as $3/4$, 4, -8, $8/9$, $1/9$, ..., bringing to discover many new hypergeometric identities. Thanks to such unusual arguments and to the inspired titles of well-known papers, these identities are often referred to as "strange identities". An unpublished but nevertheless very often cited list of strange identities due to Gosper has been a major source of testing identities for different methods conceived to solve two problems: i) to find a general method to prove conjectured identities, ii) to find "closed" forms for sums.
By defining *closed forms* as linear combinations of hypergeometric terms, the WZ method of creative telescoping [1, 10, 11] succeeds in solving problem i) and in conjunction with Petkovšek algorithm [5] solves problem ii) as well. Another approach to the problem i) that makes use of the Lagrange inversion formula (LIF for short) is presented in [2]. Here the authors prove many strange identities using, along with the LIF, special functions' factorizations as well as classic and new hypergeometric transformations. They also quote identities 6.5 and 6.6 in their paper they are not able to prove with these methods.
We think that this approach is worth to be further examinated. We are especially interested in investigating hypergeometric sums with respect to some Riordan array (R.A. for short) transformations as in [4]. Since convolution is one of the simplest R.A. transformations, see for example [8], we begin to study hypergeometric sums by interpreting them in terms of such transformations. By using the LIF and some results of classical elimination theory (see [9]) we find a closed

form for many hypergeometric sums, including the above-mentioned sums 6.5 and 6.6 in [2].
We do this in an algorithmic way. Our algorithm copes with a class of identities of the type ${}_2F_1$ with one parameter and involving convolutions of elementary functions. We aim to extend its scope to sums with more parameters and involving more general transformations. Finally, we point out that we use the system for symbolic computation Maple to compute the various steps of our algorithm.

2 The Algorithm

Let us consider the class ${}_2F_1$ of terminating hypergeometric series with one parameter and having no more than n elements, $n \in \mathbb{N}$, and investigate its elements of the form:

$$s(n) = {}_2F_1\left(\begin{matrix} a_1 n + b_1, -n \\ a_2 n + b_2 \end{matrix} \middle| \lambda \right) \tag{1}$$

with $a_1,\ a_2,\ b_1,\ b_2,\ \lambda \in \mathbb{Q}$. Our objective is to find a closed form for the sum $s(n)$. According to [6], we say that $s(n)$ is in *closed form* if it is equal to a linear combination of hypergeometric terms.
If $n \in \mathbb{N}$, let us assume that $(a_2 n + b_2)_k \neq 0,\ 0 \leq k < n$, to guarantee that the sum is well defined, and $(a_1 n + b_1)_k \neq 0,\ 0 \leq k < n$, so that the sum has exactly n non-zero terms. So, we can represent $s(n)$ as a factor (depending only on n) multiplied by the n-th term of a convolution of two hypergeometric functions. We can do this in many different ways, but we look in particular for a convolution representation which involves just "simple" hypergeometric functions, i.e. hypergeometric function that can also be represented without using the usual F-notation. Since we know that ${}_1F_0(a,\ ;\lambda t) = (1-\lambda t)^{-a}$, we can easily obtain a representation of this form. As a matter of fact, by using the well known identities

$$(-n)_k = \frac{(-1)^k\, n!}{(n-k)!},\ (an+b)_k = \frac{(-1)^k (1-b-(a+1)n)_n}{(1-b-(a+1)n)_{n-k}} \tag{2}$$

we can write:

$$s(n) = \sum_{k=0}^{n} \frac{(a_1 n + b_1)_k (-n)_k}{(a_2 n + b_2)_k k!}\ \lambda^k =$$

$$\frac{n!}{(1-b_2-(a_2+1)n)_n} \sum_{k=0}^{n} \frac{(a_1 n + b_1)_k}{k!} \lambda^k \frac{(1-b_2-(a_2+1)n)_{n-k}}{(n-k)!} \tag{3}$$

or also

$$s(n) = \frac{n!}{(1-b_2-(a_2+1)n)_n} [t^n] (1-\lambda t)^{-a_1 n - b_1} (1-t)^{b_2 - 1 + (a_2+1)n}. \tag{4}$$

We observe that, when $\lambda = 1$, $s(n)$ can be easily computed by means of Newton's rule; so, let us suppose $\lambda \neq 1$. We arbitrarily set λ^k as the argument of the first

function in (4)(see also (3)), but of course there are infinitely many ways to write this term, such as $\lambda^n(1/\lambda)^{(n-k)}$, $(2\lambda)^n(1/2)^k(1/(2\lambda))^{(n-k)}$, ... However, all these possibilities yield to the same result.
The problem of finding a closed form for (1) is thus reduced to the problem of evaluating an expression of the form

$$y(n) = [t^n]F(t)\phi(t)^n, \tag{5}$$

where:

$$F(t) = (1-\lambda t)^{-b_1}(1-t)^{b_2-1}, \;\; \phi(t) = (1-\lambda t)^{-a_1}(1-t)^{a_2+1}. \tag{6}$$

Now, if $\phi(t) \in F_0[[t]]$ then, by Lagrange Theorem (see [3]), there exists a unique formal power series $w(t) \in F_1[[t]]$ such that $w = t\phi(w)$. Moreover, if $F(t) \in F_0[[t]]$ then

$$\sum_{n=0}^{\infty} y(n)\, t^n = \frac{F(w)}{(1-t\phi'(w))} \tag{7}$$

where $y(n)$ is as in (5).
The functions $F(t)$ and $\phi(t)$ in (6) satisfy the hypothesis of Lagrange Theorem, hence by applying (7) we obtain immediately a representation of our target generating function $y(t)$. By using the notation in [4] we have:

$$y(t) = \left[\frac{F(w)}{(1-t\phi'(w))} \,\middle|\, w = t\,\phi(w) \right]. \tag{8}$$

We explicitly observe that $y(t)$ may be obtained from (8) in a straightforward way by solving the functional equation $w = t\,\phi(w)$ with respect to w and then substituting one of its solutions (the one which meets LIF hypothesis) in the expression to the left of the vertical bar. But, in many cases, this is not a practical method. It suffices that the functional equation degree exceeds two to obtain some really cumbersome and completely useless results, that is, results that may hardly be simplified even by using a symbolic calculation system as Maple. However, the major problem in such approach is the non-existence of an algorithmic formal way to solve an algebraic equation when its degree exceeds 4.
The problem of determining $y(t)$ from (8) may be approached by using the classical elimination theory [9]. So, instead of solving $w = t\phi(w)$ we substitute each occurrences of $t = w/\phi(w)$ in the first expression in (8). This can be done either using the *Gröbner bases*, or using the classical *Sylvester resultant*. We shall use this latter. As a first step, we substitute $F(w)$, $\phi(w)$ as in (6) and $t = w/\phi(w)$ in the expression (8) to obtain

$$y(t) = \left[\frac{(1-\lambda w)^{1-b_1}(1-w)^{b_2}}{(a_1-a_2)\lambda w^2 + (a_2-\lambda a_1-\lambda)w + 1} \,\middle|\, w = t\,(1-\lambda w)^{-a_1}(1-w)^{a_2+1} \right]. \tag{9}$$

Then by letting

$$\begin{cases} y = \dfrac{(1-\lambda w)^{1-b_1}(1-w)^{b_2}}{(a_1-a_2)\lambda w^2 + (a_2-\lambda a_1-\lambda)w + 1} \\ t = \dfrac{w}{(1-\lambda w)^{-a_1}(1-w)^{a_2+1}} \end{cases} \tag{10}$$

we obtain two equations depending only on w. We can easily transform equations (10) into two polynomial equations by rising them to appropriate powers, say $s,\ q$:

$$\begin{aligned} U_1(w,y) &= M_1(w)y^s - M_2(w) = 0 \\ U_2(t,w) &= M_3(w)t^q - M_4(w) = 0 \end{aligned} \tag{11}$$

$s,\ q \in \mathbb{N}$ and $M_1(w),\ M_2(w),\ M_3(w),\ M_4(w)$ are polynomials with coefficients in $\mathbb{C}$. The generating function $y(t)$ can be determined by eliminating w from the two equations (11), that is by evaluating their resultant with respect to w. Computing the resultant of two polynomials f and g with respect to a specified variable x is a standard feature in many computer algebra systems. For instance, in Maple the command `resultant(g, f, x)` does it. In the next section we present three examples of its use. But, before concluding this section we want to point out that using the LIF and the resultant is not sufficient to solve all the sums we have considered. Our method applied to certain sums leads to algebraic equations of too high degree to be solved by hand or by Maple. So by reducing them into some equivalent differential equations and these latter into some equivalent recurrence relations (by using two functions of the shareware Maple package *gfun*, see [7]), we can solve some of these sums. But in other cases, we need to apply the Petkovšek reduction algorithm to find the generating function $y(t)$ as we shall show in the next section.
The step-by-step algorithm is as follows.
Let $s(n)$ be a sum as in (1) and $a_1 = p_1/q_1$, $a_2 = p_2/q_2$, $b_1 = r_1/s_1$, $b_2 = r_2/s_2$, $p_i, r_i \in \mathbb{Z}$, $q_i,\ s_i \in \mathbb{N}$, $i = 1, 2$, and $\lambda \in \mathbb{Q}$.

1. In order to have a well defined $s(n)$, we check if the k-th Pochhammer symbol $(p_2/q_2 n + r_1/s_1)_k \neq 0$ for every $n \in \mathbb{N}$ and every integer k, $0 \leq k < n$.
   - If ($p_2/q_2 \geq 0$ and $r_2/s_2 \geq 0$) or ($p_2/q_2 \leq -1$ and $r_2/s_2 \leq 0$) then go on to step 2
   - else if ($r_2 q_2$ is not a multiple of s_2) then go on to step 2
   - else there exist n and k such that $p_2/q_2 n + r_1/s_1 + k = 0$ and the sum is not well defined;
2. By using (9) we have to build the two polynomials U_1, U_2 in (11); for the sake of simplicity let $m_1 = M_1(w)y^s$, $m_2 = M_2(w)$, $m_3 = M_3(w)t^q$, $m_4 = M_4(w)$.
   2.1 For U_1 we have:
      - if $r_1 = 0$ then $s = s_2$ else if $r_2 = 0$ then $s = s_1$ else $s = lcm(s_1, s_2)$;
      - $m_1 = y^s[(p_1/q_1 - p_2/q_2)\lambda w^2 + (p_2/q_2 - \lambda p_1/q_1 - \lambda)w + 1]^s$; $m_2 = 1$;
      - if $1 - \frac{r_1}{s_1} < 0$ then $m_1 = m_1(1-\lambda w)^{(r_1-s_1)s/s_1}$ else $m_2 = m_2(1-\lambda w)^{(s_1-r_1)s/s_1}$;
      - if $r_2 < 0$ then $m_1 = m_1(1-w)^{-r_2 s/s_2}$ else $m_2 = m_2(1-w)^{r_2 s/s_2}$
      - $U_1 = m_1 - m_2$;

   2.2 in the same way we have for U_2:
      - if $p_1 = 0$ then $q = q_2$ else if $p_2 = 0$ then $q = q_1$ else $q = lcm(q_1, q_2)$;
      - $m_3 = w^q$; $m_4 = t^q$;
      - if $p_1 > 0$ then $m_3 = m_3(1-\lambda w)^{p_1 q/q_1}$ else $m_4 = m_4(1-\lambda w)^{-p_1 q/q_1}$;
      - if $(1+p_2/q_2) < 0$ then $m_3 = m_3(1-w)^{-q(1+p_2/q_2)}$ else $m_4 = m_4(1-w)^{q(1+p_2/q_2)}$

   - $U_2 = m_3 - m_4$;
3. Now, we have to evaluate the resultant of U_1 and U_2 with respect to w. Since usually U_1 and U_2 are not very simple, we execute this step by using a Maple function called `resultant`. Nevertheless we will use the name *resultant* just to mean the mathematical operation of calculating the resultant of two polynomials. Thus:
   - $g = resultant(U_1, U_2)$;
4. Let T_d be the transformation of an algebraic equation into an equivalent differential equation g_d and let T_r be the transformation of a differential equation into an equivalent recurrence relation. So:
   - $g_d = T_d(g)$;
   - $g_r = T_r(g_d)$;
5. Now use Petkovšek algorithm to reduce recurrence order. If $y(n)$ is the n-th element of the sequence and *reduce* is the name of Petkovšek algorithm:
   - if $order(g_r) = 1$ then we go on to step 6. else $g_r = reduce(g_r)$;
   - if $order(g_r)$ is now equal to 1 then we go on to step 6., else "the sum doesn't admit a closed form expressed as hypergeometric terms"
6. Now, we have $g_r = P(n)y(n+1) + Q(n)y(n)$ with $P(n)$, $Q(n)$ polynomials with coefficients in $\mathbb{C}$, of degree d_1 and d_2 respectivily. There exist $\alpha_1, \alpha_2, \ldots, \alpha_{d_1}, \beta_1, \beta_2, \ldots, \beta_{d_2}, \mu \in \mathbb{C}$ such that:
$$\frac{y(n+1)}{y(n)} = \frac{(n-\alpha_1)(n-\alpha_2)\ldots(n-\alpha_{d_1})}{(n-\beta_1)(n-\beta_2)\ldots(n-\beta_{d_2})}\mu$$
that is
$$y(n) = \frac{(-\alpha_1)_n(-\alpha_2)_n\ldots(-\alpha_{d_1})_n}{(-\beta_1)_n(-\beta_2)_n\ldots(-\beta_{d_2})_n}\mu^n.$$
7. From (4) we conclude:
$$s(n) = \frac{n!}{(1 - r_2/s_2 - p_2/q_2 n - n)_n} y(n).$$

We will see that in many cases the first three steps are sufficient, together with the last one, for evaluating the sum. For such sums the third step gives an algebraic equation $g(t, y) = 0$ that can easily be solved with respect to $y = y(t)$. If we are able to evaluate $[t^n]y(t)$, we obtain $s(n)$ by the seventh step. If this is not the case, the other steps are needed. We explicitly recall that there exist well known algorithms to convert algebraic equation into differential equations and to convert these latter into recurrence relations. The Petkovšek algorithm is also well-known [5].

3 Applications to Strange Identities

We are now going to show how our algorithm works on three specific problems:
$${}_2F_1\left(\left.\begin{matrix} n/2+1, -n \\ 4/3 \end{matrix}\right| \frac{8}{9}\right) = \begin{cases} 0, & n \text{ odd}, \\ \frac{(\frac{1}{2})_N}{(\frac{7}{6})_N}(3)^{-N}, & n = 2N \text{ even}, \end{cases} \qquad (12)$$

$$ {}_2F_1\left(\begin{matrix}-n+1/4, -n\\ 2n+5/4\end{matrix}\,\middle|\, \frac{1}{9}\right) = \frac{\left(\frac{5}{4}\right)_{2n}}{\left(\frac{2}{3}\right)_n \left(\frac{13}{12}\right)_n}\left(\frac{2^6}{3^5}\right)^n \tag{13} $$

$$ {}_2F_1\left(\begin{matrix}-n/2-1/6, -n\\ 4/3\end{matrix}\,\middle|\, -8\right) = (-1)^{n/2}\frac{3^{3n/2}}{n+1},\ n \text{ even} \tag{14} $$

The above identities correspond to identities 3.14, 6.5 and 5.24 sin [2]. In their paper, the authors prove many identites, such as (12) and (14), by means of the factorization of the function $f(t)$

$$ f(t) = (1-t)^p\ {}_rF_s\left(\begin{matrix}\alpha_1, ..., \alpha_r\\ \beta_1, ..., \beta_s\end{matrix}\,\middle|\, A\right). $$

They evaluate the coefficient of t^n of two representations of $f(t)$ by using the *change of variable formula*, also known as *Residue Composition Theorem*, obtaining the proof of some hypergeometric identities, such as (12) and (14), but are not able to prove the identity (13). This is relevant for the topic of the present paper, because the change of variable formula is equivalent to the LIF, on which our method is based, although our approach is costructive, that is it solves problem ii) instead of i) (see the Introduction). Our method succeeds in proving the identity (13). We point out that it succeeds in proving identity 6.6 in [2] too. Of course, it is possible to prove the identities (12, 13, 14) by using the WZ method [11].

Example 1. Let us study the sum (12):

$$ s(n) = {}_2F_1\left(\begin{matrix}n/2+1, -n\\ 4/3\end{matrix}\,\middle|\, \frac{8}{9}\right) = \sum_{k=0}^{n}\frac{(n/2+1)_k(-n)_k}{(4/3)_k k!}\left(\frac{8}{9}\right)^k $$

We have $p_1 = 1$, $q_1 = 2, r_1 = 1, s_1 = 1$, $p_2 = 0$, $r_2 = 4$, $s_2 = 3$, $\lambda = 8/9$.

1. Since $p_2/q_2 = 0$ and $r_2/s_2 > 0$ we go on.
2. Polynomals construction:
   2.1 – $s = lcm(1,3) = 3$;
   – $m_1 = y^3(4/9w^2 - 4/3w + 1)^3$;
   – since $r_2 > 0$ then $m_2 = (1-w)^4$;
   – $U_1 = y^3(4/9w^2 - 4/3w + 1)^3 - (1-w)^4$;
   2.2 – $q = 2$;
   – $m_3 = w^2$; $m_4 = t^2$;
   – since $-p_1 = -1 < 0$ then $m_3 = w^2(1 - 8/9w)$;
   – since $p_2/q_2 + 1 = 1 > 0$ then $m_4 = t^2(1-w)^2$;
   – $U_2 = w^2(1 - 8/9w) - t^2(1-w)^2$;
3. We have to evaluate the resultant of U_1 and U_2 with respect to w. We use the Maple fuction `resultant` and obtain:

```
> g:=resultant(y^3*(4/9*w^2-4/3*w+1)^3-(1-w)^4,
> w^2*(1-8/9*w)-t^2*(1-w)^2,w);
```

$$ g := \frac{64}{387420489}(-9 + 9\,y^3 + 6\,y^3\,t^2 + t^4\,y^3)^3 $$

Now, by executing the other steps of the algorithm, we obtain the closed form for the sum. However steps 4. and 5. are not needed; by solving the equation $g(t, y) = 0$ with respect to y, and taking the real solution, we have:

$$y(t) = 3^{2/3}(t^2+3)^{-2/3} = {}_1F_0\left(\left.\begin{matrix}2/3\\ \end{matrix}\right| -\frac{1}{3}t^2\right).$$

Consequently:

$$y(n) = [t^n]3^{2/3}(t^2+3)^{-2/3} = \begin{cases} 0, & n \text{ odd},\\ (-1/3)^{n/2}\frac{(2/3)_{n/2}}{(n/2)!}, & n = 2N \text{ even}.\end{cases}$$

7. So we conclude:

$$s_n = n!/(-1/3-n)_n y(n)$$

and coming back to the original problem:

$${}_2F_1\left(\left.\begin{matrix}n/2+1, -n\\ 4/3\end{matrix}\right| \frac{8}{9}\right) = \begin{cases} 0, & n \text{ odd},\\ \frac{(1/2)_N}{(7/6)_N}(-3)^{(-N)}, & n = 2N \text{ even}.\end{cases}$$

In other cases we are not able to solve the algebraic equation obtained in step 3. In these situations we need some more efforts to solve the sum and this second example shows how we proceed.

Example 2. Let us consider the sum (13):

$$s(n) = {}_2F_1\left(\left.\begin{matrix}-n+1/4, -n\\ 2n+5/4\end{matrix}\right| \frac{1}{9}\right) = \sum_{k=0}^{n} \frac{(-n+1/4)_k(-n)_k}{(2n+5/4)_k k!}\left(\frac{1}{9}\right)^k$$

We have $p_1 = -1$, $q_1 = 1, r_1 = 1, s_1 = 4$, $p_2 = 2$, $q_2 = 1\, r_2 = 5$, $s_2 = 4$.

1. Since $p_2/q_2 = 2 \geq 0$ and $r_2/s_2 = 5/4 \geq 0$ the sum is well defined.
2. Polynomals construction:
   2.1 – $s = lcm(4, 4) = 4$;
   – $m_1 = y^4(-1/3w^2 + 2w + 1)^4$;
   – since $1 - r_1/s_1 = 3/4 > 0$ then $m2 := (1 - w/9)^3$;
   – since $r_2 = 5 > 0$ then $m_2 = (1 - w/9)^3(1 - w)^5$;
   – $U_1 = y^4(-1/3w^2 + 2w + 1)^4 - (1 - w/9)^3(1 - w)^5$;
   2.2 – $q = 1$;
   – $m_3 = w$; $m_4 = t$;
   – since $-p_1 = 1 > 0$ then $m_4 = t(1 - w/9)$;
   – since $p_2/q_2 + 1 = 3 > 0$ then $m_4 = t(1 - w/9)(1 - w)^3$;
   – $U_2 = w - t(1 - w/9)(1 - w)^3$;

3. We have to evaluate the resultant of U_1 and U_2 with respect to w. To do so we use the Maple function `resultant` and obtain:

```
> g:=resultant(y^4*(-w^2/3+2*w+1)^4-(1-w/9)^3*( 1-w)^5,
> w-t*(1-w/9)*(1-w)^3,w):
> g:=map(factor,collect(g,y));
```

$$g := \frac{1}{22876792454961}(64\,t+9)^8\,y^{16} - \frac{4}{847288609443}(64\,t+9)^6\,y^{12} + \frac{10}{94143178827}(64\,t+9)^4\,y^8 - \frac{4}{282429536481}(1024\,t+63)\,(64\,t+9)^2\,y^4 + + \frac{1}{387420489}$$

4. We have now to apply the transformations T_d and T_r to the equation g. We use the two functions `algeqtodiffeq` and `diffeqtorec` in the Maple package *gfun* and obtain:

```
> g_d:=algeqtodiffeq(g,y(t));
```

$$g\_d := 60\,\mathrm{y}(t) + (18+416\,t)\,\mathrm{D}(y)(t) + (192\,t^2+27\,t)\,(D^{(2)})(y)(t)$$

```
> g_r:=diffeqtorec(g_d,y(t),y(n));
```

$$g\_r := (60+224\,n+192\,n^2)\,\mathrm{y}(n) + (45\,n+18+27\,n^2)\,\mathrm{y}(n+1)$$

5. Since $order(g_r) = 1$ we go to the next step.

6. By using Maple, we get:

```
> g_r:=map(factor,g_r);
```

$$g\_r := 4\,(4\,n+3)\,(12\,n+5)\,\mathrm{y}(n) + 9\,(n+1)\,(3\,n+2)\,\mathrm{y}(n+1)$$

and consequently:

$$y(n) = {}_2F_1\left(\left.\begin{matrix}3/4,\, 5/12\\ 2/3\end{matrix}\,\right|\, -\frac{64}{9}\right)$$

7. So we have:

$$s(n) = \frac{n!}{(-1/4-3n)_n}y(n)$$

And the final result:

$$s(n) ={}_2 F_1\left(\left.\begin{matrix}-n+1/4, -n\\ 2n+5/4\end{matrix}\,\right|\,\frac{1}{9}\right) = \frac{(5/4)_{2n}}{(2/3)_n\,(13/12)_n}\left(\frac{2^6}{3^5}\right)^n.$$

The last example shows a sum for which step 4 gives a recurrence relation of order greater than 1 so that it is necessary to use Petkovšek algorithm, step 6, to reduce its order.

Example 3. Let us consider the sum (14):

$${}_2F_1\left(\left.\begin{matrix}-n/2-1/6, -n\\ 4/3\end{matrix}\,\right|\, -8\right) = \sum_{k=0}^{n}\frac{(-n/2-1/6)_k(-n)_k}{(4/3)_k k!}(-8)^k$$

We have $p_1 = -1$, $q_1 = 2, r_1 = -1, s_1 = 6$, $p_2 = 0$, $r_2 = 4\; s_2 = 3$, $\lambda = -8$.

1. Since $p_2/q_2 = 0 \geq 0$ and $r_2/s_2 = 4/3 > 0$ the sum is well defined.
2. Polynomals construction:
   2.1 – $s = lcm(6,3) = 6$;
   – $m_1 = y^6(4w^2+4w+1)^6$;
   – since $1 - r_1/s_1 = 7/6 > 0$ then $m2 = (1+8w)^7$;
   – since $r_2 = 4 > 0$ then $m_2 = (1+8w)^7(1-w)^8$;
   – $U_1 = y^6(4w^2+4w+1)^6 - (1+8w)^7(1-w)^8$;

   2.2 – $p_2 = 0$ then $q = 2$;
   – $m_3 = w^2$; $m_4 = t^2$;
   – since $-p_1 = 1 > 0$ then $m_4 = t^2(1+8w)$;
   – since $p_2/q_2 + 1 = 1 > 0$ then $m_4 = t^2(1+8w)(1-w)^2$;
   – $U_2 = t^2(1+8w) - t^2(1+8w)(1-w)^2$;
3. We have to evaluate the resultant of U_1 and U_2 with respect to w. Suppose we have already assigned U_1 and U_2 in a Maple session as in the previous step. The Maple fuction `resultant` yields:

```
> g:=resultant(U_1,U_2,w):
> g:=map(factor,collect(g,y));
```

$$g := 2097152\,t^6\,(27\,t^2+1)^{12}\,y^{18} - 2097152\,(1458\,t^6+624\,t^4+48\,t^2+1)\,(27\,t^2+1)^8\,y^{12} + 2097152\,(531441\,t^6+34992\,t^4+624\,t^2+2)\,(27\,t^2+1)^4\,y^6 - 2097152$$

4. By applying the transformation T_d to g and the transformation T_r to g_d, we obtain:

```
> g_d:=algeqtodiffeq(g,y(t));
```

$$g\_d := (84+9828\,t^2)\,\mathrm{y}(t) + (20412\,t^3+516\,t)\,\mathrm{D}(y)(t) + (3+8019\,t^4+378\,t^2)\,(D^{(2)})(y)(t) + (729\,t^5+54\,t^3+t)\,(D^{(3)})(y)(t)$$

```
> g_r:=diffeqtorec(g_d,y(t),y(n));
```

$$g\_r := (9828+13851\,n+5832\,n^2+729\,n^3)\,\mathrm{y}(n) + (1872+1758\,n+540\,n^2+54\,n^3)\,\mathrm{y}(n+2) + (12\,n^2+47\,n+60+n^3)\,\mathrm{y}(n+4)$$

5. The recurrence g_r has order 4. So, we execute the function *mriduce*, our implementation of Petkovšek algorithm. The output is the ratio $y(n+1)/y(n)$:
   – since $order(g_r) = 4$ then

```
> mriduce(g_r,y,n,4,1);
```

$$\left\{\frac{I\,\sqrt{3}\,(3\,n+4)}{n+2},\ -\frac{I\,\sqrt{3}\,(3\,n+4)}{n+2}\right\}$$

6. So we have two sequences:

$$y(n) = (4/3)_n/(2)_n 3^{3/2n}(\pm i)^n$$

They differ only for odd values of n. Therefore, for $n = 2N$ we have:

$$y(n) = (4/3)_{2N}/(2)_{2N} 3^{3N}(-1)^N.$$

7. Thus the last step yields:

$$s(n) = \frac{n!}{(-1/3-n)_n} y(n)$$

To obtain the final identity we use (2) and find:

$${}_2F_1 \left(\left. \begin{matrix} -n/2-1/6, -n \\ 4/3 \end{matrix} \right| -8 \right) = (-1)^N \frac{3^{3N}}{2N+1}, \; n = 2N \text{ even}$$

4 Conclusions

The method we propose in addition to Petkovšek algorithm solves, by using a generating functions approach, problem ii) (see Introduction) for a class of ${}_2F_1$ hypergeometric sums with one parameter. The method succeeds in solving some strange identities (6.5 e 6.6 in [2]) which, as far as we know, have not been solved by an analogous approach.

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