# Left-inversion of combinatorial sums

Cristiano Corsani, Donatella Merlini, Renzo Sprugnoli
Dipartimento di Sistemi e Informatica
Via Lombroso 6/17- Firenze
e-mail: `resp@ingfi1.ing.unifi.it`

December 19, 1996

**Abstract** The inversion of combinatorial sums is a fundamental problem in algebraic combinatorics. Some combinatorial sums, such as $a_n = \sum_k d_{n,k} b_k$, can not be inverted in terms of the orthogonality relation because the infinite, lower triangular array $P = \{d_{n,k}\}$'s diagonal elements are equal to zero (except $d_{0,0}$). Despite this, we can find a left-inverse $\bar{P}$ such that $\bar{P}P = I$ and therefore are able to left-invert the original combinatorial sum, and thus obtain $b_n = \sum_k \bar{d}_{n,k} a_k$.

**Résumé** L'inversion des sommes combinatoires est un problème fondamental dans l'algèbre combinatoire. Certaines sommes combinatoires, par exemple $a_n = \sum_k d_{n,k} b_k$, ne peuvent pas être inverties selon la rélation d'orthogonalité, parce que les éléments sur la diagonale de la matrice triangulaire inférieure $P = \{d_{n,k}\}$ sont nuls (sauf $d_{0,0}$). Malgré cela, on peut bien souvent définir une matrice left-inverse $\bar{P}$ telle que $\bar{P}P = I$ et, par conséquent , on peut left-invertir la somme combinatoire d'origine, en obtenant $b_n = \sum_k \bar{d}_{n,k} a_k$.

## 1 Introduction

The problem of inverting combinatorial sums has long interested researchers and the main reference on the subject is the famous book [11] by John Riordan "*Combinatorial Identities*". Riordan summarized his results in a paper [10], and some authors subsequently tried to give a unitary approach to his methods (see, e.g., Chu [4], Gould and Hsu [6], Egorychev [5] and Sprugnoli [15]). Some authors, such as Milne [9], have examined the problem from various other points of view; a particularly interesting approach through the Umbral Calculus has been given by Roman [12].

We aim at obtaining a substantial generalization of Riordan's results by showing that the method of generating functions, we examine in this paper, together with the concept of Riordan arrays, are powerful tools for proving a large class of inversions, that strictly includes all the inversions proposed in Riordan's book.

We are mainly interested in the set $\mathbf{R}[\![t]\!]$ of *formal power series* $f(t) = \sum_{k=0}^{\infty} f_k t^k$ having real coefficients in some indeterminate $t$; however, instead of $\mathbf{R}$, we could consider any field $\mathbf{F}$ with 0 characteristic, in particular the field $\mathbf{C}$ of complex numbers. If $+$ and $\cdot$ denote the usual sum and Cauchy product in $\mathbf{R}[\![t]\!]$, this is an integral domain; the smallest field containing $\mathbf{R}[\![t]\!]$ is the field $\mathbf{R}((t))$ of *formal Laurent power series* $f(t) = \sum_{k=m}^{\infty} f_k t^k$, with $m \in \mathbf{Z}$. The *order* of $f(t) = \sum_{k=m}^{\infty} f_k t^k$ is the minimum value of $k$ for which $f_k \neq 0$. $\mathbf{R}_s[\![t]\!]$ denotes the set of all formal power series of order $s$. In particular, $\mathbf{R}_0[\![t]\!]$ is the set of *invertible* power series, i.e., power series $f(t)$ for which $f_0 = f(0) \neq 0$: it is well-known that $g(t) \in \mathbf{R}[\![t]\!]$ such that $f(t)g(t) = 1$ exists only for these series. For a complete theory of formal power series, the reader is referred to Henrici [8].

If $\{f_k\}_{k\in\mathbf{N}}$ is a sequence of real numbers, its generating function $f(t)$ is defined as:

$$f(t) = \mathcal{G}_t\{f_k\}_{k\in\mathbf{N}} = \mathcal{G}\{f_k\} = \sum_{k=0}^{\infty} f_k t^k \in \mathbf{R}[\![\mathbf{t}]\!].$$

As usual, the notation $[t^k]$ stands for the "coefficient of" operator and, therefore, if $f(t) = \sum f_k t^k$ is a formal power series, then $[t^k]f(t) = f_k$.

The concept of a *Riordan array* is a convenient way of expressing certain infinite, lower triangular arrays $\{d_{n,k} | n, k \in \mathbf{N}, k \leq n\}$. A Riordan array is a pair $(d(t), h(t))$ of formal power series, with $d(t) \in \mathbf{R}_0[\![\mathbf{t}]\!]$; it defines an infinite, lower triangular array $\{d_{n,k}\}$ according to the rule:

$$d_{n,k} = [t^n]\, d(t)(th(t))^k. \tag{1.1}$$

The most common example of a Riordan array is the Pascal triangle, for which $d(t) = h(t) = (1-t)^{-1}$. When $h(t) \in \mathbf{R}_0[\![\mathbf{t}]\!]$ the Riordan array is called *proper* and since the diagonal elements of the corresponding $\{d_{n,k}\}$ are all different from 0, the array is invertible, and its inverse is also a proper Riordan array. No other Riordan array can be inverted in the usual row-by-column product. Proper Riordan arrays form a group called the *Riordan group*. Riordan arrays are the class of lower triangular, infinite arrays for which combinatorial sums can be expressed in terms of generating functions; more precisely, we have:

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n] d(t) f(th(t)) \tag{1.2}$$

when $f(t)$ is the generating function of the sequence $\{f_k\}_{k\in\mathbf{N}}$.

Since $\mathbf{R}[\![t]\!]$ and $\mathbf{R}[\![y]\!]$, (in which $t$ and $y$ are any two indeterminates), are isomorphic, $t$ is usually changed into $y$ or any other indeterminate, and vice versa, whenever it is convenient. Composition is another important operation in $\mathbf{R}[\![t]\!]$, and $f(g(t)) = f(t) \circ g(t) = f(y)|_{y=g(t)}$ is defined whenever $g(t) \in \mathbf{R}_s[\![t]\!]$ with $s \geq 1$ or $f(t)$ is a polynomial. If $f(t) \in \mathbf{R}_1[\![t]\!]$, then a unique $g(t) \in \mathbf{R}_1[\![t]\!]$ exist such that $f(g(t)) = g(f(t)) = t$, which is therefore the *compositional inverse* of $f(t)$. The elements of $\mathbf{R}_1[\![t]\!]$ are called *almost units* or *delta series*. The computation of the compositional inverse of a delta series leads us back to the famous, fundamental *Lagrange Inversion Theorem*, which we use in the formulation of Goulden and Jackson [7]: let $\phi(t) \in \mathbf{R}_0[\![t]\!]$; then a unique formal power series $w(t) \in \mathbf{R}_1[\![t]\!]$ exists such that $w = t\phi(w)$. Moreover:

1. If $f(t) \in \mathbf{R}(\!(t)\!)$ then:

$$[t^n]\, f(w) = \begin{cases} \frac{1}{n}[y^{n-1}]\, f'(y)\phi(y)^n & n \neq 0, n \geq \text{order}(f) \\ [y^0]\, f(y) - [y^{-1}]\, f(y)\phi(y)^{-1}\phi'(y) & n = 0 \end{cases} \tag{1.3}$$

2. If $F(t) \in \mathbf{R}[\![t]\!]$ and the sequence $\{c_n\}_{n\in\mathbf{N}}$ is defined by $c_n = [t^n]\, F(t)\phi(t)^n$, then:

$$c(t) = \sum_{k=0}^{\infty} c_n t^n = \frac{F(w)}{1 - t\phi'(w)}. \tag{1.4}$$

These formulas can be easily manipulated by introducing a particular notation. By writing:

$$f(t) = [g(y)\,|_y h_1(t,y) = h_2(t,y)]$$

we denote the function (or formal power series) of the indeterminate $t$, obtained by substituting the solution $y = y(t)$, with $y(0) = 0$, to the functional equation $h_1(t,y) = h_2(t,y)$ in $g(y)$. The following points should be emphasized:

- the bound variable $y$ in this notation can usually be deduced from the context, and we omit it as a subscript of the vertical bar. Whenever possible, equation $h_1(t,y) = h_2(t,y)$ is written as $y = h(t,y)$, thus clarifying which is the bound variable;
- obviously, we have $f(g(t)) = [f(y) |\, y = g(t)]$; besides, a convenient way of expressing the applicability conditions of the Lagrange Inversion Theorem is:

$$f(t) = f(w(t)) = [f(w) |\, w = t\phi(w)];$$

- in particular, if $\{c_k\}_{k\in\mathbf{N}}$ is a sequence defined as in point 2 in the Lagrange Inversion Theorem, then its generating function is:

$$c(t) = \mathcal{G}\{c_n\} = \left[\frac{F(w)}{1 - t\phi'(w)}\middle|\, w = t\phi(w)\right] = \left[\frac{F(w)}{1 - w\phi'(w)/\phi(w)}\middle|\, w = t\phi(w)\right].$$

After these preliminary notational remarks, we now go on to illustrate our method for inverting combinatorial sums with an example directly connected to the problems we are going to solve in our paper. Let us consider the combinatorial identity:

$$a_n = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} b_k, \tag{1.5}$$

where $\{b_k\}_{k\in\mathbf{N}}$ is a given sequence and $\{a_k\}_{k\in\mathbf{N}}$ is defined in terms of the $b_k$'s. The problem in inverting this identity is to find a relation defining the $b_k$'s in terms of the $a_k$'s. According to the Riordan array theory, identity (1.5) is related to the Riordan array $D = (1/(1-t), t/(1-t)^2)$, whose generic element can be found by means of relation (1.1):

$$d_{n,k} = [t^n]\,\frac{1}{1-t}\left(\frac{t^2}{(1-t)^2}\right)^k = \left[t^{n-2k}\right]\frac{1}{(1-t)^{2k+1}} = \binom{-2k-1}{n-2k}(-1)^{n-2k} = \binom{n}{2k}.$$

Therefore, the generating function $a(t)$ of the sequence $\{a_k\}_{k\in\mathbf{N}}$ is $a(t) = b(t^2/(1-t)^2)/(1-t)$ where $b(t)$ is the generating function of the sequence $\{b_k\}_{k\in\mathbf{N}}$. This relation can be inverted:

$$b\left(\frac{t^2}{(1-t)^2}\right) = (1-t)a(t), \qquad \text{or} \qquad b(y) = \left[(1-t)a(t)\,\middle|\, y = \frac{t^2}{(1-t)^2}\right].$$

The generic element $b_n$ can now be found by a series of computations related to the Lagrange Inversion Theorem; we find:

$$b_n = [y^n]\,b(y) = [y^n]\left[(1-t)a(t)\,\middle|\, t = y^{1/2}(1-t)\right] = \left[w^{2n}\right][(1-t)a(t)\,|\, t = w(1-t)] =$$

$$= \frac{1}{2n}\left[t^{2n-1}\right]((1-t)a'(t) - a(t))(1-t)^{2n} =$$

$$= \frac{1}{2n}\left(\sum_{k=0}^{2n-1}(-1)^{2n-1-k}\binom{2n+1}{2n-k-1}(k+1)a_{k+1} - \sum_{k=0}^{2n-1}(-1)^{2n-1-k}\binom{2n}{2n-k-1}a_k\right) =$$

$$= \frac{1}{2n}\left(\sum_{k=1}^{2n}(-1)^k\binom{2n+1}{2n-k}ka_k + \sum_{k=0}^{2n-1}(-1)^k\binom{2n+1}{2n-k}\frac{2n-k}{2n+1}a_k\right) =$$

$$= \frac{1}{2n}\left(\sum_{k=0}^{2n}(-1)^k\binom{2n+1}{2n-k}\frac{2nk+k+2n-k}{2n+1}a_k\right).$$

By performing some obvious simplifications, we eventually find:

$$b_n = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}a_k. \tag{1.6}$$

This inversion is not present in Riordan's book and the reason is fairly obvious. If we examine the infinite, lower triangular array defined by $D$ above, the diagonal elements are all zero (except $d_{0,0}$) and, therefore, the array cannot be inverted in the usual sense. In other terms, identity (1.5) cannot be associated to any orthogonal relation. On the other hand, our proof does not seem to be correct because the two identities $y = t^2/(1-t)^2$ and $t = y^{1/2}(1-t)$ are *not* equivalent. According to the formal power series theory (see Henrici [8]), when we have a functional equation $y = h(t)$ and $h(t) \in \mathbf{R}_s[\![t]\!]$ with $s > 1$, the solution $t = t(y)$ is not unique and there are exactly $s$ solutions $t_1(y), t_2(y), ..., t_s(y)$ which actually belong to $\mathbf{R}\left[\!\left[y^{1/s}\right]\!\right]$. In our example, among the various possibilities, we arbitrarily chose *one* solution and used it to apply the Lagrange Inversion Theorem. The question is whether or not our choice is justifiable.
The inversion is definitely correct and, if we define

$$P = \left\{\binom{n}{2k}\middle| n,k \in \mathbf{N}\right\} \qquad \bar{P} = \left\{(-1)^k\binom{2n}{k}\middle| n,k \in \mathbf{N}\right\},$$

we can easily check that $\bar{P}P = I$ and $P\bar{P}P = P$. $\bar{P}$ will be called the "left-inverse" array of $P$ and, strictly speaking, we should refer to the method we develop as a "left-inversion process".

We conclude this long introduction by summarizing the threefold aim of this paper: i) justifying the use of a single solution to a functional equation from a theoretical point of view, in situations like the preceding one; ii) examining the process of left-inversion; iii) giving a number of significant examples of left-inversion, to show how Riordan's results can be generalized and new inversions can be found.

## 2 Stretched Riordan arrays

In the Introduction, we defined the concept of Riordan array as developed by Shapiro et al. [13] and Sprugnoli [14]. Riordan arrays are just a concrete way to define the so-called 1-umbral calculus (see [12]) and, in fact, Riordan arrays are called "recursive matrices" by Barnabei, Brini and Nicoletti [2]. Formula (2.2) below is a version of the "transfer theorem" of umbral calculus (see Roman [12], pag. 50). What seems to be new in the present paper is the extension of umbral results to stretched arrays, a topic only occasionally considered in the literature (see Al-Salam and Versa [1] and Di Bucchianico doctoral thesis [3], two references suggested us by one referee).

We can easily show that the usual row-by-column product of two Riordan arrays is another Riordan array, and we have:

$$(d(t), h(t)) * (a(t), b(t)) = (d(t)a(th(t)), h(t)b(th(t))).$$

The Riordan array $(1, 1)$ is the identity matrix and if $(d(t), h(t))$ is proper, then its inverse $(\bar{d}(t), \bar{h}(t))$ can be computed by equating the identity matrix to the expression above; we find:

$$\bar{d}(y) = \left[d(t)^{-1}\middle| y = th(t)\right] \qquad \bar{h}(y) = \left[h(t)^{-1}\middle| y = th(t)\right]. \tag{2.1}$$

Since the Riordan array is proper, $h(t) \in \mathbf{R}_0 [\![t]\!]$ and, therefore, the functional equation $y = th(t)$, has a unique solution $t = t(y)$ and $\bar{d}(t)$, $\bar{h}(t)$ are well-defined. By means of the Lagrange Inversion Theorem, we can show that the generic element $\bar{d}_{n,k}$ of the inverse array is given by:

$$\bar{d}_{n,k} = \frac{1}{n}\left[t^{n-k}\right]\left(k - t\frac{d'(t)}{d(t)}\right)\frac{1}{d(t)h(t)^n} \qquad n > 0, \tag{2.2}$$

and $\bar{d}_{0,k} = d_0^{-1}\delta_{k,0}$, where $\delta_{k,0}$ is the Kronecker symbol. On the basis of this result, Sprugnoli [15] proposed an algorithm for proving the inversions in Riordan's book [11].

When $(d(t), h(t))$ is not proper, we can write $h(t) = h_{s-1}t^{s-1} + h_s t^s + \cdots = t^{s-1}(h_{s-1} + h_s t + h_{s+1}t^2 + \cdots) = t^{s-1}v(t)$, where $h_{s-1} \neq 0$, $s > 1$ and $v(t) \in \mathbf{R}_0 [\![t]\!]$. The corresponding numerical array is "vertically stretched", whereas proper Riordan arrays are lower triangular (see Figure 2.1). In this case, by going on to $\mathbf{R}(\!(t)\!)$, we can formally derive formulas (2.1) again, but

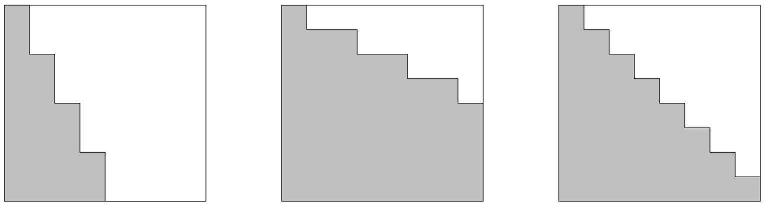

Figure 2.1: A vertically stretched R.a., a horizontally streched R.a. and a proper R.a.

we have $h(t) \in \mathbf{R}_{s-1} [\![t]\!]$ and the functional equation $y = th(t)$ no longer has a unique solution $t = t(y)$. According to the power series theory (see, e.g., Henrici [8]), $y = th(t)$ has $s$ solutions $t_1 = t_1(y), \ldots, t_s = t_s(y)$ in the following form:

$$t_j = t_j(y) = \sum_{m=1}^{\infty} \eta_m \omega_s^{jm} y^{m/s} \qquad j = 1, 2, \ldots, s.$$

Here, $\omega_s$ is any one of the $s^{th}$ primitive roots of unity. The coefficients $\eta_m$'s do not depend on $j$, i.e., they are all the same in the $s$ formal power series in $\mathbf{R}\left[\!\left[\omega_s^j y^{1/s}\right]\!\right]$. These $s$ formal power series are said to be *conjugate* to $h(t)$. They are well-known thanks to the multisectioning series theory (see, e.g., Riordan [11]). Their main property is that:

$$\frac{1}{s}\sum_{j=1}^{s} t_j(y) \in \mathbf{R}_r [\![y]\!] \qquad r > 0,$$

i.e., if we make the average of all of them, we obtain a formal power series in which the roots of unity and the fractional powers of $y$ disappear.

Properly speaking, formulas (2.1) now correspond to $s$ pairs of functions, one for each choice of $t_j(y)$, $j = 1, 2, \ldots, s$. Let us denote the pair obtained considering the $j$-th solution $t_j(y)$ by $\left(\bar{d}^{[j]}(y), \bar{h}^{[j]}(y)\right)$, $j = 1, 2, \ldots, s$. Since $d(t) \in \mathbf{R}_0 [\![t]\!]$, $\bar{d}^{[j]}(y)$ is well-defined and belongs to $\mathbf{R}_0\left[\!\left[\omega_s^j y^{1/s}\right]\!\right]$ for every $j$. As far as $\bar{h}^{[j]}(y)$ is concerned, let us make the following remarks. If we write $h(t) = t^{s-1}v(t)$, with $v(t) \in \mathbf{R}_0 [\![t]\!]$ an invertible formal power series, and then fix any $j$ between 1 and $s$, we should have $y = t_j(y)^s v(t_j(y))$ or $t_j(y)^s = yv(t_j(y))^{-1}$, where $v(t_j(y))^{-1}$ is well-defined in

$\mathbf{R}\left[\!\left[\omega_s^j y^{1/s}\right]\!\right]$. On the other hand, by means of the previous definition, we have:

$$\bar{h}^{[j]}(y) = h(t_j(y))^{-1} = \frac{t_j(y)}{t_j(y)^s} v(t_j(y))^{-1} = \frac{t_j(y)}{y},$$

thus establishing a very simple relation between $\bar{h}^{[j]}(y)$ and the solution $t_j(y)$ to the basic functional equation. This also shows that $\bar{h}^{[j]}(y)$ is well-defined and belongs to $\mathbf{R}_{1-s}\left(\!\left(\omega_s^j y^{1/s}\right)\!\right)$.

It is worth noting that in our introductory example, by solving the functional equation, we obtain:

$$t_1(y) = \frac{y^{1/2}}{1+y^{1/2}} \qquad t_2(y) = -\frac{y^{1/2}}{1-y^{1/2}}.$$

From these expressions and (2.2), we can easily find the two pairs of the inverse Riordan array:

$$d_1(y) = \frac{1}{1+y^{1/2}} \qquad d_2(y) = \frac{1}{1-y^{1/2}}$$

$$h_1(y) = \frac{1}{y^{1/2}(1+y^{1/2})} \qquad h_2(y) = \frac{-1}{y^{1/2}(1-y^{1/2})}$$

From a theoretical point of view, this may be satisfactory since we obtain a good definition of the "inverse" non-proper Riordan array. However, from a practical (and numerical) point of view, the question is to establish which array corresponds to the $s$-uples of formal power series pairs. We can prove the following results:

**Theorem 2.1** *The formal power series*

$$\bar{d}_k(y) = \frac{1}{s}\sum_{j=1}^{s} \bar{d}^{[j]}(y)\left(y\bar{h}^{[j]}(y)\right)^k$$

*belong to* $\mathbf{R}[\![y]\!]$ *(properly, to* $\mathbf{R}_r[\![y]\!]$*, with* $r = \lceil k/s \rceil \quad \forall k \in \mathbf{N}$*) and, therefore, they can be taken as the column generating functions of an infinite array* $\bar{D} = \{\bar{d}_{n,k}\}_{n,k\in\mathbf{N}}$*, in which the rows generating functions have order* $ns+1$*.*

**Proof:** As previously mentioned, all the solutions $t_1(y), t_2(y), ..., t_s(y)$ of the functional equation $y = th(t)$ have the same coefficients $\eta_m$. According to Henrici [8], these coefficients can be computed by solving the equation $z = (th(t))^{1/s} = tv(t)^{1/s}$, which satisfies normal conditions for series inversion and, in particular, those required for applying the Lagrange Inversion Theorem. If $t(z)$ is the unique solution to the modified functional equation, then we have:

$$t_j = t_j(y) = t\left(\omega_s^j y^{1/s}\right) \qquad j = 1, 2, ..., s$$

i.e., the actual $s$ solutions are formed by simply substituting the indeterminate $y$ for the new "indeterminate" $\omega_s^j y^{1/s}$. We can associate, to the solution $t(z)$ of the modified equation, the following two functions

$$\bar{d}(z) = \frac{1}{d(t(z))} \in \mathbf{R}_0[\![z]\!] \qquad \bar{h}(z) = \frac{1}{h(t(z))} \in \mathbf{R}_{1-s}(\!(z)\!), \tag{2.3}$$

having the property that:

$$\bar{d}^{[j]}(y) = \bar{d}\left(\omega_s^j y^{1/s}\right) \qquad \bar{h}^{[j]}(y) = \bar{h}\left(\omega_s^j y^{1/s}\right) \quad j = 1, 2, ..., s.$$

This immediately shows that all the $\bar{d}^{[j]}(y)$ and the $\bar{h}^{[j]}(y)$ , $j = 1, 2, ..., s$, have the same coefficients in $\mathbf{R}\left[\!\left[\omega_s^j y^{1/s}\right]\!\right]$. The same is true for the powers $\left(y\bar{h}^{[j]}(y)\right)^k$ and all their combinations. When we make the average of these functions with $j = 1, 2, ..., s$, by a well-known property of the roots of unity, the non integer powers disappear and the integer powers all have the same coefficient, which therefore becomes the coefficient in $\bar{d}_k(y)$. Finally, from $h(t) \in \mathbf{R}_{s-1}[\![t]\!]$, some computations yield that the order of $\bar{d}_k(y)$ is $\lceil k/s \rceil$. Figure 2.1 illustrates the original array $D$, the newly defined array $\bar{D}$ and a proper Riordan array. ■

We call the non proper Riordan array, i.e., the Riordan array with $h(t) \notin \mathbf{R}_0[\![t]\!]$ *vertically stretched*, and the array $\{\bar{d}_{n,k}\}_{n,k\in\mathbf{N}}$,

$$\bar{d}_{n,k} = [y^n]\frac{1}{s}\sum_{j=1}^{s} \bar{d}^{[j]}(y)\left(y\bar{h}^{[j]}(y)\right)^k, \tag{2.4}$$

defined by means of a set of conjugate pairs of formal power series in $\mathbf{R}\left[\!\left[\omega_s^j y^{1/s}\right]\!\right]$ *horizontally stretched*, because of its shape.

**Theorem 2.2** *If we consider the row-by-column product, we find $\bar{D}D = I$ and therefore $D\bar{D}D = D$. In this sense, $\bar{D}$ is the* left-inverse *of the array $D$ and can be used to "invert" combinatorial sums related to non-proper Riordan arrays.*

**Proof:** We have:

$$\sum_k \bar{d}_{n,k}d_{k,m} = \sum_k \left([y^n]\,\frac{1}{s}\sum_{j=1}^{s} \bar{d}^{[j]}(y)\left(y\bar{h}^{[j]}(y)\right)^k\right)\left(\left[t^k\right]d(t)\,(th(t))^m\right) =$$

$$= [y^n]\,\frac{1}{s}\sum_{j=1}^{s} \bar{d}^{[j]}(y)\sum_k\left(y\bar{h}^{[j]}(y)\right)^k\left[t^k\right]d(t)\,(th(t))^m =$$

$$= [y^n]\,\frac{1}{s}\sum_{j=1}^{s} \bar{d}^{[j]}(y)d\left(y\bar{h}^{[j]}(y)\right)\left(y\bar{h}^{[j]}(y)h\left(y\bar{h}^{[j]}(y)\right)\right)^m =$$

$$= [y^n]\,\frac{1}{s}\sum_{j=1}^{s}\left[\left[d(t)^{-1}d(t)(th(t))^m\,\middle|\,z = (th(t))^{1/s}\right]\,\middle|\,z = \omega_s^j y^{1/s}\right] =$$

$$= [y^n]\,\frac{1}{s}\sum_{j=1}^{s}\left(\omega_s^{sj}y\right)^m = [y^n]\,y^m = \delta_{n,m}$$

where we freely apply the definition of $\bar{d}^{[j]}(y)$, rule $t_j(y) = y\bar{h}^{[j]}(y)$, and the above remarks concerning the evaluation of the $\eta_m$ coefficients. ■

From a practical point of view, instead of averaging on $j = 1, 2, ..., s$, we can take any $\bar{d}^{[j]}(y)$ and its corresponding $\bar{h}^{[j]}(y)$ and ignore the non integer exponents to obtain the following definition:

$$\bar{d}_{n,k} = [y^n]\,\bar{d}^{[j]}(y)\left(y\bar{h}^{[j]}(y)\right)^k. \tag{2.5}$$

In our introductory example, by using $j = 1$, we have:

$$\bar{d}_{n,k} = [y^n]\,\frac{1}{1+\sqrt{y}}\left(\frac{\sqrt{y}}{1+\sqrt{y}}\right)^k = \left[y^{n-k/2}\right]\frac{1}{(1+\sqrt{y})^{k+1}} =$$

$$= \left[z^{2n-k}\right] \frac{1}{(1+z)^{k+1}} = \binom{-k-1}{2n-k} = \binom{k+1+2n-k-1}{2n-k}(-1)^{2n-k} = (-1)^k \binom{2n}{k}.$$

Alternatively, we can use the solution to the modified functional equation $y = (th(t))^{1/s}$, the functions $\bar{d}(y)$ and $\bar{h}(y)$ as defined in (2.3) and relation (2.1), and obtain the following definition of $\bar{d}_{n,k}$:

$$\bar{d}_{n,k} = [y^n]\bar{d}_k(y) = [y^{sn}]\, \bar{d}(y) \left(y^s \bar{h}(y)\right)^k . \tag{2.6}$$

The reader can easily prove that this definition gives the same results as the previous example, if $\bar{d}(y) = (1+y)^{-1}$ and $\bar{h}(y) = 1/(y(1+y))$. Formula (2.6) is perhaps the most direct method, while (2.4) and (2.5) are more elegant and show that the $\bar{d}_k(y)$ can be defined as belonging to $\mathbf{R}\,[\![y]\!]$ and as relating to the series multisectioning.

**Remark**: In general, to extract the coefficient of $y^n$ from a function $f(t)$ where $t$ is any one of the $s^{th}$ solutions to the functional equation $y = t^s v(t)$, $v(t) \in \mathbf{R}_0\,[\![y]\!]$, we can extract the coefficient of $y^{sn}$ from $f(t)$ being $t$ the unique solution to the functional equation $y = tv(t)^{1/s}$.

We can now prove a formula for $\bar{d}_{n,k}$, which generalizes formula (2.2) for proper Riordan arrays:

**Theorem 2.3** *(s-transfer formula)*

$$\bar{d}_{n,k} = \frac{1}{sn}\left[t^{(sn-k)}\right]\left(k - t\frac{d'(t)}{d(t)}\right)\frac{1}{d(t)v(t)^n} \qquad n > 0, \qquad \bar{d}_{0,k} = \delta_{k,0}/d_0.$$

**Proof:** We have:

$$\bar{d}_{n,k} = [y^{sn}]\, \bar{d}(y) \left(y^s \bar{h}(y)\right)^k = [y^{sn}]\left[\left.\frac{t^k}{d(t)}\right| y = (th(t))^{1/s}\right] =$$

$$= [y^{sn}]\left[\left.\frac{t^k}{d(t)}\right| y = tv(t)^{1/s}\right] = \frac{1}{sn}\left[t^{sn-1}\right]\frac{d}{dt}\left(\frac{t^k}{d(t)}\right)\frac{1}{v(t)^n} =$$

$$= \frac{1}{sn}\left[t^{sn-1}\right]\frac{kt^{k-1}d(t) - t^k d'(t)}{d(t)^2}\frac{1}{v(t)^n}$$

and we therefore conclude that:

$$\bar{d}_{n,k} = \frac{1}{sn}\left[t^{sn-k}\right]\left(k - t\frac{d'(t)}{d(t)}\right)\frac{1}{d(t)v(t)^n} \qquad n > 0.$$

When $n = 0$, we obtain:

$$\bar{d}_{0,k} = \left[y^0\right]\left[\left.\frac{t^k}{d(t)}\right| t = yv(t)^{-1/s}\right] = \left[t^0\right]\frac{t^k}{d(t)} - \left[t^{-1}\right]\frac{t^k v(t)^{1/s}\left(v(t)^{-1/s}\right)}{d(t)} =$$

$$= \left[t^{-k}\right]\frac{1}{d(t)} + \frac{1}{s}\left[t^{-1}\right]\frac{t^k v(t)^{-1}v'(t)}{d(t)} = \left[t^{-k}\right] d(t)^{-1}$$

or $\bar{d}_{0,k} = \delta_{k,0}/d_0$, because $v(t), d(t) \in \mathbf{R}_0\,[\![t]\!]$.

These formulas solve, from a theoretical point of view, the problem of inverting combinatorial sums involving Riordan arrays $\{d_{n,k}\}$, with $h(t) \in \mathbf{R}_s\,[\![t]\!]$ and $s > 0$, since every sum

$$a_n = \sum_{k=0}^{\lfloor n/s \rfloor} d_{n,k} b_k \qquad \text{has the inverse} \qquad b_n = \sum_{k=0}^{ns} \bar{d}_{n,k} a_k.$$

It is actually often more convenient to apply the Lagrange Inversion Theorem to obtain the inverse formula directly, and this is illustrated in detail in the next section. As a result of the previous remarks, we can generalize the Lagrange Inversion Theorem in the following way:

**Theorem 2.4** *Let $h(t) = t^{s-1}v(t) \in \mathbf{R}_{s-1}[\![t]\!]$ and set $\phi(t) = v(t)^{-1/s}$; consequently a unique formal power series $w(t) \in \mathbf{R}[\![t]\!]$ exists such that $w = t\phi(w)$. Moreover if $f(t) \in \mathbf{R}((t))$ then:*

$$[t^n]\left[f(w)\middle| t = w^s v(w)\right] = \begin{cases} \frac{1}{ns}\left[y^{ns-1}\right] f'(y)v(y)^{-n} & n \neq 0 \\ \left[y^0\right] f(y) + \frac{1}{s}\left[y^{-1}\right] f(y)v(y)^{-1}v'(y) & n = 0; \end{cases} \tag{2.7}$$

*if $F(t) \in \mathbf{R}[\![t]\!]$ then:*

$$[t^n]\left[\frac{F(w)}{1 - w\phi'(w)/\phi(w)}\middle| t = w^s v(w)\right] = [y^{ns}]\frac{F(y)}{v(y)^n}. \tag{2.8}$$

**Proof:** If $f(t) \in \mathbf{R}((t))$ then, by using the previous remark and the classical Lagrange Inversion Theorem, we obtain:

$$[t^n]\left[f(w)\middle| t = w^s v(w)\right] = [y^{ns}]\left[f(w)\middle| w = y\phi(w)\right] =$$

$$= \begin{cases} \frac{1}{ns}\left[y^{ns-1}\right] f'(y)v(y)^{-n} & n \neq 0 \\ \left[y^0\right] f(y) + \frac{1}{s}\left[y^{-1}\right] f(y)v(y)^{-1}v'(y) & n = 0 \end{cases}$$

If $F(t) \in \mathbf{R}[\![t]\!]$ then, by using the same approach as before, we have:

$$[t^n]\left[\frac{F(w)}{1 - w\phi'(w)/\phi(w)}\middle| t = w^s v(w)\right] = [y^{ns}]\left[\frac{F(w)}{1 - w\phi'(w)/\phi(w)}\middle| w = y\phi(w)\right] =$$

$$= [y^{ns}]\left[\frac{F(w)}{1 - y\phi'(w)}\middle| w = y\phi(w)\right] = [y^{ns}]\frac{F(y)}{v(y)^n}.$$

■

## 3 Sample inversions

In the simplest cases, formula (2.3) is a very direct method of inverting (or left-inverting) combinatorial sums. In the Introduction, we presented an example developed in a rather natural way corresponding to the general algorithm we are going to present below. Formula (2.3) can also be applied with $s = 2, d(t) = (1-t)^{-1}$ and $v(t) = (1-t)^{-2}$:

$$\bar{d}_{n,k} = \frac{1}{2n}[t^{2n-k}]\left(k - t\frac{1-t}{(1-t)^2}\right)(1-t)(1-t)^{2n} =$$

$$= \frac{k}{2n}[t^{2n-k}](1-t)^{2n+1} - \frac{1}{2n}[t^{2n-k-1}](1-t)^{2n} = \frac{k}{2n}\binom{2n+1}{2n-k}(-1)^k - \frac{1}{2n}\binom{2n}{2n-k-1}(-1)^{k-1} =$$

$$= (-1)^k\left(\frac{k}{2n}\binom{2n+1}{k+1} + \frac{1}{2n}\binom{2n}{k+1}\right) = \binom{2n}{k}(-1)^k\left(\frac{k(2n+1)}{2n(k+1)} + \frac{2n-k}{2n(k+1)}\right) = \binom{2n}{k}(-1)^k.$$

We can prove the more general left-inversion ($s \in \mathbf{Z}^+$) in the same way:

$$a_n = \sum_{k=0}^{\lfloor n/s \rfloor} \binom{n}{sk} b_k \qquad\qquad b_n = \sum_{k=0}^{sn} (-1)^{sn-k} \binom{sn}{k} a_k$$

and the reader can check the following "rotated" version of the same left-inversion:

$$a_n = \sum_{k=0}^{\lfloor n/s \rfloor} \binom{n+p}{sk+p} b_k \qquad\qquad b_n = \sum_{k=0}^{sn} (-1)^{sn-k} \binom{sn+p}{k+p} a_k$$

The generalizations of Riordan inversions related to the Abel identity are not so elegant as the left inversions which only involve binomial coefficients. By using exponential generating functions, we find that the sum related to the Riordan array $D = (e^{xt}, t^{s-1}e^{qt})$ is:

$$\frac{a_n}{n!} = \sum_{k=0}^{\lfloor n/s \rfloor} \frac{(x+qk)^{n-sk}}{(n-sk)!} \frac{b_k}{k!},$$

and, when $s > 1$, this identity cannot be expressed simply in terms of binomial coefficients but formula (2.3) can be used for inverting it. Since $v(t) = e^{qt}$, we obtain:

$$\bar{d}_{n,k} = \frac{1}{sn}[t^{sn-k}](k - tx)\frac{1}{e^{xt}e^{nqt}} = (-1)^{sn-k}\left(\frac{k}{sn}\frac{(x+nq)^{sn-k}}{(sn-k)!} + \frac{x}{sn}\frac{(x+nq)^{sn-k-1}}{(sn-k-1)!}\right) =$$

$$= \left(\frac{k(x+nq)}{sn} + \frac{x(sn-k)}{sn}\right)(-1)^{sn-k}\frac{(x+qn)^{sn-k-1}}{(sn-k)!} = \frac{kq+xs}{s}(-1)^{sn-k}\frac{(x+qn)^{sn-k-1}}{(sn-k)!}.$$

Therefore, the left-inverse relation is:

$$\frac{b_n}{n!} = \sum_{k=0}^{ns} (-1)^{sn-k} \frac{kq+xs}{s} \frac{(x+qn)^{sn-k-1}}{(sn-k)!} \frac{a_k}{k!}.$$

Not all cases are so linear and we often have to make multiple use of the Lagrange Inversion Theorem, especially when the original sum contains some factor depending on $n$ (the bound variable in the sum that has to be inverted). The Riordan array approach still applies, but an initial application of the Lagrange Inversion Theorem is required in order to obtain the Riordan array to be used in the left-inversion process. The following algorithm can then be applied both in difficult and very simple cases (see the Introduction).

**Algorithm**: Let the identity $a_n = \sum_k d_{n,k} b_k$ be given:

**(1)** Put the sum into a suitable form for a Riordan array approach (see [15] for a discussion on this point);

**(2)** Express $a_n$ as $[t^n]G(t)$

  **(2a)** if $G(t)$ does not depend on $n$, $G(t) = a(t)$ is the generating function of the sequence $\{a_n\}_{n \in \mathbf{N}}$; then proceed with step (3);

  **(2b)** else, use the Lagrange Inversion Theorem (2.8) to find the generating function $a(t)$;

**(3)** invert the identity obtained in step (2);

**(3a)** if $h(t) = 1$, simply apply the Riordan array rule (1.2) backwards;

**(3b)** if $y = th(t)$ can be solved explicitly, then substitute the solution in the inverse relation and apply the Riordan array rule (1.2) backwards;

**(3c)** otherwise, to obtain the expression for $b_n$ in terms of $a_k$'s, use the (2.7) form of the Lagrange Inversion Theorem, if possible, utilizing the notations in the previous section.

**(3d)** as an alternative, apply the (2.8) form of the Lagrange Inversion Theorem backwards.

Let us use this algorithm for solving the following left-inversion problem. Given the identity:

$$a_n = \sum_{k=0}^{\lfloor n/s \rfloor} \left( \binom{p + qk - k}{n - sk} + \xi \binom{p + qk - k}{n - sk - 1} \right) b_k,$$

we want to find out the value of $\xi$ corresponding to the inverse identity's simplest form. The sum is clearly related to the Riordan array $D = ((1+t)^p, t^{s-1}(1+t)^{q-1})$ and steps (1),(2) of the Algorithm give us:

$$a_n = [t^n](1+t)^p b\left(t^s(1+t)^{q-1}\right) + \xi [t^{n-1}](1+t)^p b\left(t^s(1+t)^{q-1}\right) =$$

$$= [t^n](1+\xi t)(1+t)^p b\left(t^s(1+t)^{q-1}\right).$$

In this case, $G(t)$ directly gives the generating function $a(t)$, and we can invert the relation by step (3) and obtain

$$b\left(t^s(1+t)^{q-1}\right) = \frac{a(t)}{(1+\xi t)(1+t)^p}. \tag{3.1}$$

This expression can be written in the following way:

$$b(y) = \left[ \frac{a(t)}{(1+\xi t)(1+t)^p} \,\middle|\, y = t^s(1+t)^{q-1} \right] = \left[ \frac{a(t)}{(1+\xi t)(1+t)^p} \,\middle|\, t = \frac{y^{1/s}}{(1+t)^{(q-1)/s}} \right]. \tag{3.2}$$

According to the results in the previous section, we can substitute a new indeterminate $x$ for $y^{1/s}$ without having to worry about the multiple solutions of the functional equation $y = t^s(1+t)^{q-1}$. In order to apply step (3c) of the Algorithm, we should differentiate the right hand side of (3.1) and thus obtain a rather complicated expression. Instead, let us try to apply step (3d), since the last expression in (3.2) could be obtained by an application of the Lagrange Inversion Theorem. In fact, if we apply formula (1.4) to $F(t) = a(t)(1+t)^{-1-p}$ and $\phi(t) = (1+w)^{-(q-1)/s}$, we find the following generating function:

$$\left[ \frac{F(w)}{1 - w\phi'(w)/\phi(w)} \,\middle|\, w = t\phi(w) \right] = \left[ \frac{a(w)}{\left(1 + \frac{s+q-1}{s} w\right)(1+w)^p} \,\middle|\, w = \frac{t}{(1+w)^{(q-1)/s}} \right].$$

However, this generating function is $b(y)$ if we change variables and set $\xi = (s+q-1)/s$. Therefore:

$$b_n = [t^{sn}] \frac{a(t)}{(1+t)^{p+1+n(q-1)}} = \sum_{k=0}^{ns} \binom{-p-1-n(q-1)}{ns-k} a_k = \sum_{k=0}^{ns} \binom{p + n(q-1+s) - k}{ns - k} (-1)^{ns-k} a_k$$

and this is the left-inverse identity we were looking for. This example generalizes inversion 2 in Table 2.2 (Gould Class of Inverse Relations) in Riordan [11], which is obtained by setting $s = 1$.

The following example is very simple but illustrates an application of step (3c). Let us start with the identity:

$$a_n = \sum_{k=0}^{\lfloor n/s \rfloor} \binom{n-1+k}{n-sk} b_k.$$

By step (2) of the Algorithm, we can write $a_n = [t^n](1+t)^{n-1} b\left(t^s(1+t)\right)$. Since $G(t)$ depends on $n$, we apply the Lagrange Inversion Theorem to find the generating function $a(t)$:

$$a(t) = \left[ \left. \frac{(1+w)^{-1} b(w^s(1+w))}{1-\frac{w}{1+w}} \right| w = t(1+w) \right] = \left[ b\left(w^s(1+w)\right) \middle| w = t(1+w) \right].$$

We can now invert this relation:

$$b(t) = \left[ a\left(\frac{w}{1+w}\right) \middle| w = \frac{t^{1/s}}{(1+w)^{1/s}} \right]$$

and apply the Lagrange Inversion Theorem to it:

$$b_n = \frac{1}{sn}[w^{sn-1}] \frac{1}{(1+w)^2} a'\left(\frac{w}{1+w}\right) \frac{1}{(1+w)^n} = \frac{1}{sn} \sum_{k=0}^{ns-1} \binom{sn}{sn-k-1} (-1)^{sn-k-1}(k+1) a_{k+1}.$$

We can now change the variable $k$ into $k-1$ and obtain the following left-inverse relation:

$$b_n = \frac{1}{sn} \sum_{k=0}^{sn} k \binom{sn}{sn-k} (-1)^{sn-k} a_k,$$

valid for $n > 0$. For $n = 0$, we obviously have $b_0 = a_0$.

The next example is related to the Abel identity and generalizes inversion 4a in Table 5.1 of Riordan's book. We start with the identity (in the following formulas a "hat" denotes exponential generating functions):

$$\frac{a_n}{n!} = \sum_{k=0}^{\lfloor n/s \rfloor} \left( \frac{(x+n+k)^{n-sk}}{(n-sk)!} - \frac{(x+n+k)^{n-sk-1}}{(n-sk-1)!} \right) \frac{b_k}{k!} =$$

$$= [t^n] e^{(x+n)t} \widehat{b}(t^s e^t) - [t^{n-1}] e^{(x+n)t} \widehat{b}(t^s e^t) = [t^n](1-t) e^{(x+n)t} \widehat{b}(t^s e^t).$$

We have thus performed step (2) of the Algorithm. In this case, however, $G(t)$ depends on $n$ and cannot be considered the generating function of the sequence on the left. We therefore apply step (2b) by using the Lagrange Inversion Theorem with $F(t) = e^{xt}(1-t)\widehat{b}(t^s e^t)$ and $\phi(t) = e^t$:

$$\widehat{a}(t) = \left[ \left. \frac{(1-w) e^{xw} \widehat{b}(w^s e^w)}{1 - w e^{-w} e^w} \right| w = t e^w \right] = \left[ e^{xw} \widehat{b}(w^s e^w) \middle| w = t e^w \right].$$

By step (3), we now have $\widehat{b}(w^s e^w) = \widehat{a}(w e^{-w}) e^{-xw}$, or:

$$\widehat{b}(t) = \left[ \widehat{a}(w e^{-w}) e^{-xw} \,|\, t = w^s e^w \right] = \left[ \widehat{a}(w e^{-w}) e^{-xw} \middle| w = \frac{t^{1/s}}{e^{w/s}} \right].$$

We can try to apply the Lagrange Inversion Theorem backwards. If we set $F(w) = \left(1 + \frac{w}{s}\right) \widehat{a}(w e^{-w}) e^{-xw}$ and $\phi(w) = e^{-w/s}$, the element $[x^{ns}] F(x) \phi(x)^{ns}$ has the following generating function:

$$\frac{F(w)}{1 - t\phi'(w)} = \left[ \left. \frac{\left(1+\frac{w}{s}\right) \widehat{a}(w e^{-w}) e^{-xw}}{1 - w e^{-w/s}(-\frac{1}{s}) e^{w/s}} \right| w = \frac{x}{e^{w/s}} \right] = \left[ \widehat{a}(w e^{-w}) e^{-xw} \middle| w = \frac{x}{e^{w/s}} \right] = \widehat{b}(t)$$

when $x = t^{1/s}$. Therefore, we have:

$$\frac{b_n}{n!} = [t^{ns}]\widehat{b}(t) = [w^{ns}]e^{-xw}\widehat{a}(we^{-w})e^{-nw} + \frac{1}{s}[w^{ns-1}]e^{-xw}\widehat{a}(we^{-w})e^{-nw} =$$

$$= \sum_{k=0}^{ns}(-1)^{ns-k}\frac{(x+n+k)^{ns-k}}{(ns-k)!}\frac{a_k}{k!} + \sum_{k=0}^{ns-1}\frac{(-1)^{ns-k-1}}{s}\frac{(x+n+k)^{ns-k-1}}{(ns-k-1)!}\frac{a_k}{k!}.$$

The last sum obtained can be extended to $k = ns$ by multiplying and dividing by $(ns-k)!$; finally, some simple manipulations yield:

$$\frac{b_n}{n!} = \sum_{k=0}^{ns}(-1)^{ns-k}\frac{sx+sk}{s}\frac{(x+n+k)^{ns-k-1}}{(ns-k)!}\frac{a_k}{k!}.$$

This is the left-inverse relation we were looking for; when we set $s = 1$, it obviously coincides with inversion 4a in Table 3.1 of Riordan's book [11].

# References


[1] Al-Salam and Versa. Generalized Sheffer polymomials. *Duke Math. Journal*, 37:361–365, 1970.

[2] M. Barnabei, A. Brini, and G. Nicoletti. Recursive matrices and umbral calculus. *Journal of Algebra*, 75:546–573, 1982.

[3] A. Di Bucchianico. *Polynomials of convolution type*. Doctoral thesis, Rijksuniversiteit Groningen, 1991.

[4] W. Chu. Inversion techniques and combinatorial identities: strange evaluation of hypergeometric series. *Pure Mathematics and Applications*, 4(4):409–427, 1993.

[5] G.P. Egorychev. Integral representation and the computation of combinatorial sums. In *Translations of mathematical monographs 59*. AMS, 1984.

[6] H. W. Gould and L. C. Hsu. Some new inverse series relations. *Duke Math. J.*, 859–891, 1973.

[7] I. P. Goulden and D. M. Jackson. *Combinatorial Enumeration*. John Wiley & Sons, 1983.

[8] P. Henrici. *Applied and computational complex analysis*. Wiley, New York, 1991.

[9] S. Milne. Inversion properties of triangular arrays of numbers. *Analysis*, 1:1–7, 1981.

[10] J. Riordan. Inverse relations and combinatorial identities. *Am. Math. Mon.*, 71:485–498, 1964.

[11] J. Riordan. *Combinatorial identities*. Wiley, New York, 1968.

[12] S. Roman. *The Umbral Calculus*. Academic Press, Orlando, 1978.

[13] L. V. Shapiro, S. Getu, W.-J. Woan, and L. Woodson. The Riordan group. *Discrete Applied Matematics*, 34:229–239, 1991.

[14] R. Sprugnoli. Riordan arrays and combinatorial sums. *Discr. Math.*, 132:267–290, 1994.

[15] R. Sprugnoli. A unitary approach to combinatorial inversions. Technical Report 14, Dipartimento di Sistemi e Informatica, Università di Firenze, 1994.