An algebraic approach to a class of combinatorial sums

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Abstract

We obtain a theorem that allows us to find the generating function for some combinatorial sums related to non proper Riordan arrays. This function can be used to obtain a closed form for the sum (or an asymptotic evaluation). We give several examples to illustrate some practical applications of the theorem.

1 Introduction

A Riordan array $D = \{d_{n,k}\}_{n,k\in N}$ is an infinite lower triangular array characterized by a pair $(d(t), h(t))$ of formal power series. This concept was formally introduced under the name of Riordan array by Shapiro et al. in 1991 (see e.g. [13]), but a similar concept can be found in previous works, such as [1, 2, 7, 8, 9, 10, 11, 12]. In [14], Riordan arrays are used as a valid tool for solving combinatorial sums, i.e., for finding a closed form or an asymptotic evaluation by means of generating functions. More specifically, Riordan arrays allow us to translate a sum $\sum_{k=0}^{\infty} d_{n,k} f_k$ into a transformation of the generating function $f(t)$ for the sequence $\{f_k\}_{k\in N}$ and the pair of formal power series $d(t)$ and $h(t)$ defining the Riordan array D.

By applying the Lagrange Inversion Formula (see [4]), it is possible to extend the class of combinatorial sums that can be solved by means of Riordan arrays. In Section 2, we begin by introducing some notations and refer to a result illustrated in [14], where a sum $\sum_{k=0}^{\infty} d_{n,k} \hat{f}_k^{[h]}$ with $d_{n,k}$ as above and $\hat{f}_k^{[h]}$ coefficients of the function $\hat{f}^{[h]}(y) = f(t)_{|y=th(t)} = \sum_{k=0}^{\infty} \hat{f}_k^{[h]} y^k$, is transformed into an expression involving $f(t)$ and the function $d(t)$ of the Riordan array. This transformation allows us to prove the classical identities of Abel and Gould and many other interesting combinatorial sums in a very simple way but it only holds if $h(t)$ is a formal power series such that $h(0) \neq 0$.

This transformation is so important that, in Section 3, we extend it to functions $h(t)$ having $h(0) = 0$; in particular, we prove Theorem 3.3 which generalizes the results of [14] and constitutes this paper's most important result. Finally, in Section 4, we provide some applications of this formula that are mostly related to some particular classes of numbers such as Catalan, Motzkin, Schröder and Stirling numbers.

The present technics can be considered as an alternative to existing methods such as the "snake oil" by Wilf [16] and the Zeilberger's algorithm [17].

2 Riordan arrays and combinatorial sums

Let $\{f_k\}_{k\in N}$ be a sequence of (real) numbers. The *generating function* $f(t)$ of the sequence is defined as $f(t) = \mathcal{G}\{f_k\} = \sum_{k=0}^{\infty} f_k t^k$. The notation $[t^n]f(t)$ denotes the *coefficient* of t^n in the Taylor development of $f(t)$ around $t = 0$. A *Riordan array* (see Sprugnoli [14]) is an infinite lower triangular array $\{d_{n,k}\}_{n,k\in N}$ defined by a pair $(d(t), h(t))$ of formal power series in the sense that $d_{n,k} = [t^n]d(t)(th(t))^k$. We usually assume that $d(0) \neq 0$; when we also have $h(0) \neq 0$, the Riordan array is called *proper*. $\mathcal{F}_r$ having $r \geq 0$ denotes the set of formal power series whose first non-zero element is in position r. In this paper, we write $[f(y) \mid y = g(t)]$ instead of the more traditional $f(y)_{|y=g(t)}$ ($g(0) = 0$); as usual, $\delta_{n,m}$ denotes the Kronecker delta.

When dealing with combinatorial sums, Riordan arrays are a very powerful tool, as can be seen in the following theorem, (see, e.g., [14]):

Theorem 2.1 *Let $D = (d(t), h(t))$ be a Riordan array and let $f(t)$ be the generating function of the sequence $\{f_n\}_{n\in N}$. Then:*

$$\sum_{k=0}^{\infty} d_{n,k} f_k = [t^n]d(t)f(th(t)).$$

Conversely, if $\{d_{n,k} | n, k \in N\}$ is an infinite lower triangular array such that for every sequence $\{f_k\}_{k\in N}$ we have:

$$\mathcal{G}\left\{\sum_{k=0}^{\infty} d_{n,k} f_k\right\} = d(t)f(th(t)),$$

where $f(t)$ is the generating function of the sequence $\{f_k\}_{k\in N}$, and $d(t)$, $h(t)$ are two formal power series not depending on $f(t)$, then the triangle defined by the Riordan array $D = (d(t), h(t))$ coincides with $\{d_{n,k}\}$.

If $f(t)$ and $h(t)$ are formal power series and $h(t) \in \mathcal{F}_0$, we can define the following function:

$$\hat{f}^{[h;0]}(y) = \left[f(t) \,\middle|\, y = th(t)\right]. \tag{2.1}$$

Since $h(t) \in \mathcal{F}_0$, the Lagrange Inversion Formula assures us that the functional equation $y = th(t)$ has only one solution in a neighbourhood of $t = 0$, and thus $\hat{f}^{[h;0]}(y)$ exists and is uniquely determined. The Lagrange Inversion Formula also allows us to compute the coefficients $\hat{f}_k^{[h;0]}$ of the series $\hat{f}^{[h;0]}(y) = \sum_{k=0}^{\infty} \hat{f}_k^{[h;0]} y^k$ in terms of $f(t)$ and $h(t)$. Consequently, we obtain $\hat{f}_0^{[h;0]} = f_0$, and:

$$\hat{f}_k^{[h;0]} = [y^k]\hat{f}^{[h;0]}(y) = [y^k]\left[f(t) \,\middle|\, y = th(t)\right] = \frac{1}{k}[t^{k-1}]\frac{f'(t)}{h(t)^k}, \qquad k \neq 0.$$

Example 2.1 Let us consider $f(t) = (1-t)^{-1}$ and $h(t) = (1-t)^{-2}$. By solving the equation $th(t) = y$ for t, we find:

$$t(y) = \frac{1 + 2y - \sqrt{1+4y}}{2y}.$$

Thus we have:

$$\hat{f}^{[h;0]}(y) = \left[\frac{1}{1-t} \middle| t = \frac{1 + 2y - \sqrt{1+4y}}{2y} \right] = \frac{1 + \sqrt{1+4y}}{2}.$$

Moreover:

$$\hat{f}_k^{[h;0]} = \frac{1}{k}[t^{k-1}]\frac{f'(t)}{h(t)^k} = \frac{1}{k}[t^{k-1}](1-t)^{2k-2} = \frac{(-1)^{k-1}}{k}\binom{2(k-1)}{k-1}.$$

■

The function $\hat{f}^{[h;0]}(y)$ satisfies the following very important property (which we base our results on):

Lemma 2.2 *If* $\hat{f}^{[h;0]}(y) = \left[f(t) \,\middle|\, y = th(t) \right]$ *and* $h(t) \in \mathcal{F}_0$, *then*

$$\hat{f}^{[h;0]}(th(t)) = f(t). \tag{2.2}$$

Proof: We have:

$$\hat{f}^{[h;0]}(th(t)) = [[f(y)|\, z = yh(y)]|\, z = th(t)] = [f(y)|\, th(t) = yh(y)].$$

But the equation $th(t) = yh(y)$, having $h(t) \in \mathcal{F}_0$, only has one solution $y(t) = t$ in a neighbourhood of $t = 0$, and so we obtain our proof. ■

It is easy to verify that the relation (2.2) holds for the functions $f(t)$ and $h(t)$ in Example 2.1. Let us now consider a proper Riordan array $D = (d(t), h(t))$ and the sum $\sum_{k=0}^{\infty} d_{n,k}\hat{f}_k^{[h;0]}$. We obtain the following result (see also [15]) from Theorem 2.1 and Lemma 2.2:

Theorem 2.3 *Let* $D = (d(t), h(t))$ *be a proper Riordan array and* $f(t)$ *a formal power series, then*

$$\sum_{k=0}^{\infty} d_{n,k}\hat{f}_k^{[h;0]} = [t^n]d(t)f(t). \tag{2.3}$$

■

Example 2.2 Let D be the proper Riordan array defined by the functions $d(t) = (1-t)^{-1}$ and $h(t) = (1-t)^{-2}$, and let $f(t) = (1-t)^{-1}$. We point out that $h(t)$ and $f(t)$ are the same functions we discussed in Example 2.1. We easily find:

$$d_{n,k} = [t^n]\frac{1}{1-t}\frac{t^k}{(1-t)^{2k}} = \binom{n+k}{2k}.$$

By applying Theorem 2.3, we then get the following closed formula:

$$1 + \sum_{k=1}^{\infty}\binom{n+k}{2k}\binom{2(k-1)}{k-1}\frac{(-1)^{k-1}}{k} = [t^n]\frac{1}{(1-t)^2} = n+1.$$

■

3 The extended method

Theorem 2.3 is a very important tool in computing combinatorial sums, and some other more significant examples of its applications are presented in Sprugnoli [14]. In this section, we generalize formula (2.3) to case $h(t) \in \mathcal{F}_{s-1}$ with $s \geq 1$, while in the next section, we illustrate some of the formula's applications to some well-known classes of numbers.

When $s > 1$, the equation $y = th(t)$ has exactly s solutions $t_1, t_2, \ldots, t_s$ in a neighbourhood of $t = 0$ (see, e.g., Henrici [6]); these solutions have the following form:

$$t_j = \sum_{m=1}^{\infty} \eta_m \omega_s^{jm} y^{m/s} \qquad\qquad j = 1, \ldots, s,$$

where

$$\eta_m = \frac{1}{m}\{[x^{m-1}](1+Q(x))^{-m/s}\}(h_{s-1})^{-m/s}$$

and

$$Q(x) = h_{s-1}^{-1}(h_s x + h_{s+1}x^2 + \cdots) \qquad \omega_s = e^{\frac{2\pi i}{s}} \qquad (i = \sqrt{-1}).$$

Let $f(t) = \sum_{k=0}^{\infty} f_k t^k$ be a formal power series; then for all $j = 1, \ldots, s$, we can denote the composition $f(t_j(y))$ as $\sum_{k=0}^{\infty} \lambda_k \omega_s^{jk} y^{k/s}$, where λ_k only depends on η_k and f_k. Each function $f(t_j(y))$ is not a formal power series but their sum $\frac{1}{s}\sum_{j=1}^{s} f(t_j(y))$ is, in fact:

$$\frac{1}{s}\sum_{j=1}^{s} f(t_j(y)) = \frac{1}{s}\sum_{j=1}^{s}\sum_{k=0}^{\infty} \lambda_k \omega_s^{jk} y^{k/s} = \frac{1}{s}\sum_{k=0}^{\infty} \lambda_k \sum_{j=1}^{s} \omega_s^{jk} y^{k/s} =$$

$$= \frac{1}{s}\sum_{m=0}^{\infty} s\lambda_{ms} y^m = \sum_{m=0}^{\infty} \lambda_{ms} y^m,$$

because $\sum_{j=1}^{s} \omega_s^{jk} = \begin{cases} s & \text{if } k = ms \\ 0 & \text{otherwise} \end{cases}.$

We can therefore define the formal power series:

$$\hat{f}^{[h;s-1]}(y) = \frac{1}{s}\sum_{j=1}^{s} f(t_j(y)), \tag{3.1}$$

where $t_j(y)h(t_j(y)) = y$ for $j = 1, \ldots, s$. A similar approach is proposed in [3].

Example 3.1 Let us consider $h(t) = t/(1-t)^{-2} \in \mathcal{F}_1$ (with $s = 2$) and $f(t) = (1-t)^p$, with real p. By solving the equation $th(t) = y$ for t, we obtain the following two values:

$$t_1(y) = \frac{\sqrt{y}}{1+\sqrt{y}}, \qquad t_2(y) = \frac{-\sqrt{y}}{1-\sqrt{y}}.$$

Consequently, we have:

$$\hat{f}^{[h;1]}(y) = \frac{(1-t_1(y))^p + (1-t_2(y))^p}{2} = \frac{1}{2}\left[\frac{1}{(1+\sqrt{y})^p} + \frac{1}{(1-\sqrt{y})^p}\right] = \frac{g(\sqrt{y}) + g(-\sqrt{y})}{2},$$

with $g(y) = (1+y)^{-p} = \sum_{n=0}^{\infty} \binom{p+n-1}{n} (-y)^n$. Finally, by applying the well-known bisection formula, we obtain:

$$\hat{f}^{[h;1]}(y) = \sum_{n=0}^{\infty} g_{2n} y^n = \sum_{n=0}^{\infty} \binom{2n+p-1}{2n} y^n.$$

■

Let us now evaluate $\hat{f}^{[h;s-1]}(th(t)) = \frac{1}{s}\sum_{j=1}^{s} f(t_j(th(t)))$: we begin by evaluating $t_j(th(t))$. For all integers $j = 1, \ldots, s$, $t_j(z)$ is a solution to the functional equation $z = th(t)$ and so $y_j(t) = t_j(th(t))$ is a solution to the equation

$$yh(y) = th(t). \tag{3.2}$$

This, in turn, has exactly s solutions in a neighbourhood of $t = 0$ when $h(t)$ belongs to $\mathcal{F}_{s-1}$ (see Henrici [6], Th. 2.4f and Cor. 2.4g). Then, by solving equation 3.2, we get the s solutions $y_j(t)$, $j = 1, \ldots, s$, such that:

$$\hat{f}^{[h;s-1]}(th(t)) = \frac{1}{s}\sum_{j=1}^{s} f(y_j(t)). \tag{3.3}$$

In Example 3.1, we obtained exactly s solutions to the equation 3.2. Unfortunately, there are some functions $h(t)$ for which the solution to the equation 3.2, from a formal point of view, produces a number $n \geq s$ of functions. Therefore, the problem is to characterize the s solutions we are looking for. A very trivial check-over can be made by substituting the n functions in the equation 3.2 and by then discarding the $n-s$ ones that don't verify the equation. In the following theorem, we propose an alternative method for selecting the s solutions $y_j(t)$ from the n functions :

Theorem 3.1 *Let $h(t) = t^{s-1}g(t) = t^{s-1}(h_{s-1} + h_s t + h_{s+1}t^2 + \ldots)$ be a formal power series belonging to $\mathcal{F}_{s-1}$, with $s \geq 1$; the functional equation $yh(y) = th(t)$ has s solutions $y_j(t)$, $j = 1, \ldots, s$ in a neighbourhood of $t = 0$. The solutions have the following properties:*

- *$y_j(t)$ is a formal power series belonging to $\mathcal{F}_1$;*
- *$y_j(t) = \eta_1 \omega_s^j h_{s-1}^{1/s} t + O(t^2)$ where $\eta_1^s h_{s-1} = 1$, i.e., $\eta_1 \omega_s^j h_{s-1}^{1/s}$ is a s^{th} root of unity.*

Proof: For all $j = 1, \ldots, s$, $y_j(t) = t_j(th(t)) = \sum_{m=1}^{\infty} \eta_m \omega_s^{jm} (th(t))^{m/s}$, and $(t_j(y))^s g(t_j(y)) = y$; since $t_j(0) = 0$ and $(t_j(y))^s = \eta_1^s y + O(y^2)$, we have:

$$g(t_j(y)) = \frac{y}{\eta_1^s y + O(y^2)}, \qquad g(0) = \frac{1}{\eta_1^s} = h_{s-1}.$$

Then, if $t_j(y)$ is a solution of $y = th(t)$, we have the following condition:

$$\eta_1^s h_{s-1} = 1. \tag{3.4}$$

Let us now consider $y_j(t)$: by setting $g(t) = h_{s-1}(1 + G(t))$, with $G(0) \neq 0$, we obtain:

$$y_j(t) = \sum_{m=1}^{\infty} \eta_m \omega_s^{jm} (t^s g(t))^{m/s} = \sum_{m=1}^{\infty} \eta_m \omega_s^{jm} h_{s-1}^{m/s} t^m ((1+G(t))^{1/s})^m.$$

The previous expression and condition (3.4) prove the second part of the theorem. By the binomial theorem we also find:

$$(1+G(t))^{1/s} = \sum_{k=0}^{\infty} \binom{1/s}{k} G(t)^k,$$

and, since $G(t) \in \mathcal{F}_1$, it follows that $(1+G(t))^{1/s}$ is a formal power series belonging to $\mathcal{F}_0$. This concludes our proof. ■

Example 3.2 Let us consider $h(t) = t^2/\sqrt{1+t^3} \in \mathcal{F}_2$; by solving the equation:

$$\frac{t^3}{\sqrt{1+t^3}} = \frac{y^3}{\sqrt{1+y^3}},$$

we obtain the following six functions:

$$f_1(t) = -\frac{t}{\sqrt[3]{1+t^3}}, \qquad f_2(t) = -\frac{\left(-1+i\sqrt{3}\right)t}{2\sqrt[3]{1+t^3}}, \qquad f_3(t) = \frac{\left(1+i\sqrt{3}\right)t}{2\sqrt[3]{1+t^3}},$$

$$f_4(t) = t, \qquad f_5(t) = \frac{\left(-1+i\sqrt{3}\right)t}{2}, \qquad f_6(t) = -\frac{\left(1+i\sqrt{3}\right)t}{2}.$$

By developing the first three functions in a Mac Laurin series, we find:

$$f_1(t) = -t + \frac{1}{3}t^4 + O\left(t^7\right),$$

$$f_2(t) = \left(\frac{1-i\sqrt{3}}{2}\right)t - \left(\frac{1-i\sqrt{3}}{6}\right)t^4 + O\left(t^7\right), \quad f_3(t) = \left(\frac{1+i\sqrt{3}}{2}\right)t - \left(\frac{1+i\sqrt{3}}{6}\right)t^4 + O\left(t^7\right).$$

We have $f_1(0) = f_2(0) = f_3(0) = 0$, but their first coefficient is not a cubic root of unity. Conversely, the three polynomials $f_4(t)$, $f_5(t)$ and $f_6(t)$ are the formal power series desired. ■

We can extract the coefficients $\hat{f}_k^{[h;s-1]}$ from $\hat{f}^{[h;s-1]}(y)$ by using a generalization of the Lagrange Inversion Formula (see [3]). This formula can be proven in the same way as the traditional one (see Goulden and Jackson [4]).

Theorem 3.2 *Let $f(t)$ and $h(t) = t^{s-1}g(t) \in \mathcal{F}_{s-1}$ be two formal power series; the coefficients $\hat{f}_k^{[h;s-1]}$ of the function $\hat{f}^{[h;s-1]}(y)$ defined in formula (3.1) can be computed as follows:*

$$\hat{f}_k^{[h;s-1]} = \begin{cases} f_0 & k = 0 \\ \dfrac{1}{ks}[t^{sk-1}]\dfrac{f'(t)}{g(t)^k} & k > 0. \end{cases}$$

■

We are now ready to generalize Theorem 2.3 to the case $h(t) \in \mathcal{F}_{s-1}$:

Theorem 3.3 *Let $D = (d(t), h(t))$ be a Riordan array with $h(t) \in \mathcal{F}_{s-1}$ and $f(t)$ be a formal power series; then:*

$$\sum_{k=0}^{\infty} d_{n,k}\hat{f}_k^{[h;s-1]} = \frac{1}{s}[t^n]d(t)\sum_{j=1}^{s} f(y_j(t)) \tag{3.5}$$

where $y_j(t)h(y_j(t)) = th(t)$ for all $j = 1, \ldots, s$.

Proof: The proof directly follows from Theorem 2.1 and formula (3.3). ■

4 Applications

The following are some examples of combinatorial sums treated by Theorem 3.3:

Example 4.1 Let us consider the Riordan array

$$D = \left(\frac{1}{(1-t)^p}, \frac{t}{(1-t)^2}\right)$$

and the function $f(t) = (1-t)^p$. By definition, we find:

$$d_{n,k} = [t^n]\frac{1}{(1-t)^p}\left(\frac{t^2}{(1-t)^2}\right)^k = \binom{n+p-1}{2k+p-1},$$

and from Theorem 3.2:

$$\hat{f}_k^{[h;1]} = \frac{1}{2k}[t^{2k-1}]\frac{f'(t)}{g(t)^k} = \frac{p}{p+2k}\binom{p+2k}{2k}.$$

By solving the equation $th(t) = yh(y)$, i.e. $t^2/(1-t)^2 = y^2/(1-y)^2$, we find:

$$y_1(t) = t, \qquad y_2(t) = \frac{-t}{1-2t}.$$

By making some evaluations, we get:

$$\hat{f}^{[h;1]}(th(t)) = \frac{f(y_1(t)) + f(y_2(t))}{2} = \frac{(1-t)^p}{2}\left(1 + \frac{1}{(1-2t)^p}\right).$$

Finally, by applying Theorem 3.3, we obtain the following identity:

$$\sum_{k=0}^{\infty}\binom{n+p-1}{2k+p-1}\frac{p}{p+2k}\binom{p+2k}{2k} = \sum_{k=0}^{\infty} d_{n,k}\hat{f}_k^{[h;1]} = [t^n]d(t)\hat{f}^{[h;1]}(th(t)) =$$

$$= [t^n]\frac{1}{2}\left(1 + \frac{1}{(1-2t)^p}\right) = \frac{1}{2}\delta_{n,0} + \frac{1}{2}\binom{-p}{n}(-2)^n = \frac{1}{2}\delta_{n,0} + \binom{p+n-1}{n}2^{n-1}.$$

■

Example 4.2 Let us now consider the Riordan array

$$D = \left(1-t, \frac{t}{(1-t)^3}\right) \quad \text{or} \quad d_{n,k} = \binom{n+k-2}{n-2k}$$

and the function $f(t) = 1/(1-t)$. By applying Theorem 3.2 to them, we get:

$$\hat{f}_k^{[h;1]} = -\frac{1}{2k}\binom{3k-2}{k-1}, \quad \hat{f}_0^{[h;1]} = 1,$$

and, by solving the equation $t^2/(1-t)^3 = y^2/(1-y)^3$ for y, we obtain:

$$y_1(t) = t, \quad y_2(t) = \frac{3t-1+(1-t)\sqrt{1-4t}}{2t^2}, \quad y_3(t) = \frac{3t-1-(1-t)\sqrt{1-4t}}{2t^2}.$$

Since the third expression is not a formal power series, we deduce that the correct solutions are $y_1(t)$ and $y_2(t)$. After some simplifying, we find:

$$\hat{f}^{[h;1]}(th(t)) = \frac{3-2t+\sqrt{1-4t}}{4(1-t)}.$$

The coefficient $[t^n]d(t)\hat{f}^{[h;1]}(th(t))$ is related to the Catalan numbers, defined as $C_n = [t^n](1-\sqrt{1-4t})/(2t)$, as follows:

$$d_{n,0}\hat{f}_0^{[h;1]} - \sum_{k=1}^{\infty} \binom{n+k-2}{n-2k} \frac{1}{2k} \binom{3k-2}{k-1} = \delta_{n,0} - \delta_{n,1} - \sum_{k=1}^{\infty} \binom{n+k-2}{n-2k} \frac{1}{2k} \binom{3k-2}{k-1} =$$

$$= -\frac{1}{2}[t^{n-1}]\frac{1-\sqrt{1-4t}}{2t} - [t^n]\frac{t-2}{2} = -\frac{1}{2n}\binom{2(n-1)}{n-1} - \frac{1}{2}\left(\delta_{n,1} - 2\delta_{n,0}\right).$$

Therefore, for $n > 0$, the following identity holds:

$$\sum_{k=1}^{\infty} \binom{n+k-2}{n-2k} \frac{1}{k} \binom{3k-2}{k-1} = C_{n-1} - \delta_{n,1}. \tag{4.1}$$

■

Example 4.3 In this example, we examine the following Riordan array:

$$D = \left(1-t, \frac{t}{(1-t)^4}\right) \quad \text{or} \quad d_{n,k} = \binom{n+2k-2}{n-2k},$$

and the same function $f(t)$ as before. We easily obtain

$$\hat{f}_k^{[h;1]} = -\frac{1}{2k}\binom{2(2k-1)}{2k-1}, \qquad \hat{f}_0^{[h;1]} = 1, \qquad \hat{f}^{[h;1]}(th(t)) = \frac{t-3-\sqrt{t^2-6t+1}}{4(t-1)}.$$

This time, $\hat{f}^{[h;1]}(th(t))$ is related to Schröder's numbers $S_n = [t^n](1+t-\sqrt{t^2-6t+1})/(4t)$:

$$d_{n,0}\hat{f}_0^{[h;1]} - \sum_{k=1}^{\infty} \binom{n+2k-2}{n-2k} \frac{1}{2k} \binom{4k-2}{2k-1} = \delta_{n,0} - \delta_{n,1} - \sum_{k=1}^{\infty} \binom{n+2k-2}{n-2k} \frac{1}{2k} \binom{4k-2}{2k-1} =$$

$$= -[t^n]\frac{-3+t-\sqrt{1-6t+t^2}}{4} = \delta_{n,0} - [t^{n-1}]\frac{1+t-\sqrt{1-6t+t^2}}{4t} = \delta_{n,0} - S_{n-1}.$$

Therefore, the following identity holds:

$$\sum_{k=1}^{\infty} \binom{n+2k-2}{n-2k} \frac{1}{2k} \binom{2(2k-1)}{2k-1} = S_{n-1} - \delta_{n,1}. \tag{4.2}$$

■

Example 4.4 This example is related to the Motzkin numbers defined as $M_n = [t^n](1-t-\sqrt{1-2t-3t^2})/(2t^2)$. We consider the Riordan array:

$$D = (1, t(1+t)) \quad \text{or} \quad d_{n,k} = \binom{n-3k-1}{n-2k}(-1)^n$$

and the function $f(t) = (1+t)$. Among the solutions to the equation $th(t) = yh(y)$, we have to take $y_1(t) = t$ and $y_2(t) = (-1-t-\sqrt{1-2t-3t^2})/2$. Moreover, we note that $h(t) = tf(t)$ and this yields:

$$f(y_2(t)) = \frac{t^2 f(t)}{y_2(t)^2}, \qquad \hat{f}^{[h;1]}(th(t)) = \frac{f(t)}{2}\left(1 + \frac{t^2}{y_2(t)^2}\right).$$

Since $\hat{f}_k^{[h;1]} = -\binom{3k-2}{2k-1}$, by some simplifying, we obtain:

$$\delta_{n,0} - \sum_{k=1}^{\infty}\binom{n-3k-1}{n-2k}\frac{(-1)^n}{2k}\binom{3k-2}{2k-1} = \delta_{n,0} - [t^{n-2}]\frac{1-t-\sqrt{1-2t-3t^2}}{4t^2} = \delta_{n,0} - \frac{1}{2}M_{n-2}$$

and the following identity is proved:

$$\sum_{k=1}^{\infty}\binom{n-3k-1}{n-2k}\frac{1}{k}\binom{3k-2}{2k-1} = (-1)^n M_{n-2}. \tag{4.3}$$

■

Example 4.5 Let us examine the Riordan array D having $h(t) \in \mathcal{F}_1$ and the function $f(t)$ defined by:

$$D = \left(\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^p, t\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^2\right), \qquad f(t) = \frac{1}{t}\ln\frac{1}{1-t}.$$

We can note that:

$$d_{n,k} = [t^n]\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^p t^{2k}\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^{2k} = [t^{n+p}]\left(\ln\frac{1}{1-t}\right)^{p+2k} = \frac{(p+2k)!}{(n+p)!}\begin{bmatrix} n+p \\ p+2k \end{bmatrix},$$

where $\begin{bmatrix} n \\ m \end{bmatrix}$ is a Stirling number of the first kind which satisfies:

$$\left(\ln\frac{1}{1-t}\right)^m = \sum_{n=0}^{\infty}\frac{m!}{n!}\begin{bmatrix} n \\ m \end{bmatrix} t^n,$$

(see Graham, Knuth and Patashnik [5]). Moreover, by some simplifying, we have $\hat{f}_0^{[h;1]} = f_0$ and, for $k > 0$:

$$\hat{f}_k^{[h;1]} = \frac{1}{2k}[t^{2k-1}]\frac{f'(t)}{g(t)^k} = \frac{1}{2k}[t^{2k}]\left(\frac{1}{(1-t)\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^{2k}} - \frac{1}{\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^{2k-1}}\right). \tag{4.4}$$

It is not very easy to extract the previous coefficient and we do it by examining one term at a time. By differentiating, we obtain the following for $m>1$:

$$[t^k]\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^{-m+1} = \frac{1}{k}[t^{k-1}]\frac{d}{dt}\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^{-m+1} =$$

$$= \frac{m-1}{k}[t^k]\left(\frac{1}{t}\ln\frac{1}{1-t}\right) - \frac{m-1}{k}[t^k]\frac{1}{1-t}\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^{-m},$$

and thus:

$$[t^k]\frac{1}{1-t}\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^{-m} = \frac{m-k-1}{m-1}[t^k]\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^{-m+1}.$$

We use this result to transform the first term of (4.4) into an expression involving the second one. We then examine the generating function of the Stirling polynomials (see, e.g., [5], p. 258):

$$\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^{x} = x\sum_{j=0}^{\infty}\sigma_j(x+j)t^j,$$

and, by setting $x=-2k+1$ in it, we immediately deduce that:

$$\hat{f}_k^{[h;1]} = \frac{1}{2k}\left[\sigma_{2k}(1)-(1-2k)\sigma_{2k}(1)\right] = \sigma_{2k}(1).$$

By solving the equation $\ln^2(1-t)=\ln^2(1-y)$, we obtain solutions $y_1(t)=t$ and $y_2(t)=t/(t-1)$ and this yields:

$$\hat{f}^{[h;1]}(th(t)) = \frac{-\ln(1-t)}{t} + \frac{\ln(1-t)}{2}.$$

Then, by applying Theorem 3.3, we have:

$$\sum_{k=0}^{\infty}\frac{(p+2k)!}{(n+p)!}\begin{bmatrix}n+p\\p+2k\end{bmatrix}\sigma_{2k}(1) = [t^n]\left(\frac{1}{t}\ln\frac{1}{1-t}\right)^{p}\left(\frac{1}{t}\ln\frac{1}{1-t} - \frac{t}{2}\frac{1}{t}\ln\frac{1}{1-t}\right) =$$

$$= \frac{(p+1)!}{(n+p+1)!}\begin{bmatrix}n+p+1\\p+1\end{bmatrix} - \frac{(p+1)!}{2(n+p)!}\begin{bmatrix}n+p\\p+1\end{bmatrix}.$$

■

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