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Disentangling Systematic and Idiosyncratic Dynamics in Panels of Volatility Measures

Matteo Barigozzi, Christian T. Brownlees, Giampiero M. Gallo, David Veredas

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DISENTANGLING SYSTEMATIC AND IDIOSYNCRATIC DYNAMICS IN PANELS OF VOLATILITY MEASURES

Matteo BARIGOZZI[1], Christian BROWNLEES[2], Giampiero M. GALLO[3] and David VEREDAS[4]

Abstract

Realized volatilities measured on several assets exhibit a common secular trend and some idiosyncratic pattern. We accommodate such an empirical regularity extending the class of Multiplicative Error Models (MEMs) to a model where the common trend is estimated nonparametrically while the idiosyncratic dynamics are assumed to follow univariate MEMs. Estimation theory based on seminonparametric methods is developed for this class of models for large cross-sections and large time dimensions. The methodology is illustrated using two panels of realized volatility measures between 2001 and 2008: the SPDR Sectoral Indices of the S&P500 and the constituents of the S&P100. Results show that the shape of the common volatility trend captures the overall level of risk in the market and that the idiosyncratic dynamics have an heterogeneous degree of persistence around the trend. An out–of–sample forecasting exercise shows that the proposed methodology improves volatility prediction over a number of benchmark specifications.

Keywords: Vector Multiplicative Error Model, Seminonparametric Estimation, Volatility.

JEL classification: C32, C51, G01.

[1]London School of Economics and Political Science – Department of Statistics; e-mail: m.barigozzi@lse.ac.uk.

[2]Universitat Pompeu Fabra – Department of Economics and Business & Barcelona GSE; e-mail: christian.brownlees@upf.edu.

[3]Università di Firenze – Dipartimento di Statistica, Informatica, Applicazioni; e-mail: gallog@disia.unifi.it.

[4]ECARES – Solvay Brussels School of Economics and Management – Université libre de Bruxelles; e-mail: david.veredas@ulb.ac.be.

Corresponding address: David Veredas, ECARES – Solvay Brussels School of Economics and Management – Université libre de Bruxelles, 50 Av F.D. Roosevelt CP114/04, B1050 Brussels, Belgium. Phone: +32(0)26504218. Fax: +32(0)26504475.

1 Introduction

The model suggested in this paper is motivated by the analysis of panels of realized volatility measures (RV). Visual inspection of the data shows that RVs tend to oscillate around a common average level which can be associated with the overall level of risk in the market. Disentangling systematic and idiosyncratic components allows us to understand which movements are due to common and individual sources. The memory of the idiosyncratic processes gives also interesting insights on how long individual shocks persist.

From a statistical perspective, we extend the class of Multiplicative Error Models (MEM) for nonnegative multivariate time series (Engle (2002), Engle and Gallo (2006)), and dynamic models with slowly moving components as in, *inter alia*, Engle and Rangel (2008). Such a model could also be applied to disentangle systematic and idiosyncratic dynamics in panels of market activity, risk or liquidity measures (e.g. traded volumes, spreads, trading intensities) exhibiting similar empirical regularities.

We introduce a vector MEM that decomposes the conditional expectation of each series as the product of a systematic trend (modeled as a nonparametric curve) and an idiosyncratic dynamic component modeled as univariate MEMs. A simple estimation approach makes the model appealing even when the number of series in the panel is large.

We focus on modelling realized variances alone and not the realized covariance in that for several applications, such as variance derivatives trading, volatilities are the main object of interest. The construction and modelling of large realized covariance matrices still poses some theoretical and practical challenges, and is a topic of active research (Chiriac and Voev (2011), Noureldin *et al.* (2012a), Bauwens and Storti (2013)). Asset pricing models developed in finance also motivate decomposing the risk of an assent in a systematic and idiosyncratic components. However, in these models systematic and idiosyncratic components are additive while here we adopt a multiplicative framework motivated by the stylized facts of the data.

The estimation approach developed for this class of models combines ideas from the literature on profile likelihood and copulas. First, building up on the inference from the marginals framework of Joe (1997) and Joe (2005), the joint conditional likelihood of the model is decomposed in the contribution of the marginal densities and joint copula dependence. Second, the marginal densities are used to estimate both the nonparametric common trend and the parametric idiosyncratic dynamics. We justify this approach using results from profile likelihood maximization (Staniswalis (1987) and Staniswalis (1989), Severini and Wong (1992) and Veredas *et al.* (2007)), a technique that allows to establish efficiency bounds in a seminonparametric setting. The estimation procedure boils down to a nonparametric estimation of the systematic component and the univariate estimation of the idiosyncratic marginal dynamics of each series. The large sample properties of the estimators are derived and, in particular, we show that the asymptotic variance of the estimated parameters is the smallest possible, given the seminonparametric and two–step nature of the procedure. The theory is developed in a setting that allows both large cross-sections and large time dimensions. A Monte Carlo study shows that in finite samples the estimators perform adequately and that standard inferential procedures behave satisfactorily.

We apply the model to two panels of daily realized volatilities spanning from January 2, 2001 to December 31, 2008. The first panel consists of the nine sectoral indices of the SPDR S&P500 index, while the second contains the ninety constituents of the S&P100 that have continuously been traded in the sample period. The datasets are related to each other in that the constituents of the S&P100 are also some of the main underlying assets of the SPDR sectoral indices. The empirical results of the two applications are consistent to one another. The estimated shape of the systematic risk is essentially the same in the two panels, and its level can be associated with the global level of uncertainty in the economy, which exhibits clear peaks at the beginning and end of the 2000s in correspondence to the dot–com bubble burst and the financial crisis. Once the systematic trend is accounted for idiosyncratic dynamics are mean reverting. Interestingly, the speed of reversion is rather heterogeneous across assets. For instance, in the SPDR panel, mean reversion is steady for Consumer Discretionary and Materials while it is much slower for the Technology and Energy. Moreover, the S&P100 panel exhibits on average more idiosyncratic dynamics in the sense of slower mean reversion than the SPDR sectors. Inspection of the idiosyncratic dynamics reveals interesting patterns, like Technology being more volatile during the dot–com bubble burst, Energy sector experiencing turmoil during the energy crisis in 2005–2006, and Financials being under distress during with the advent of the Financial crisis. Finally, an out–of–sample forecasting exercise is used to assess the predictive ability of the specification. We forecast realized volatility from 2007 to the end of the sample using a number of MEM–based specifications. Results show that forecasts based on the our model are able to improve the out–of–sample predictive ability in the majority of cases.

Different strands of literature relate to our work. Starting from the contribution of Engle and Rangel (2008), there has been interest in capturing secular trends in financial volatility. Among others, the list of contributions in a univariate setting includes Amado and Teräsvirta (2008), Engle *et al.* (2009) and Brownlees and Gallo (2010). Feng (2006), Rangel and Engle (2012), Hafner and Linton (2010), Long *et al.* (2011) and Colacito *et al.* (2011) extend these ideas in a multivariate setting. The paper relates also to the literature on multivariate extensions of the MEM model, like the works of Cipollini *et al.* (2006) and Hautsch (2008). Moreover, there is a long tradition of decomposing panels of financial time series into a common and an idiosyncratic component in econometrics, namely in additive conditional heteroskedastic factor models (see, among others, Diebold and Nerlove (1989), Sentana (1998) Alessi *et al.* (2009), and Gagliardini and Gourieroux (2009)). The paper builds on the realized volatility literature developed by, among others, Andersen *et al.* (2003), Aït-Sahalia *et al.* (2005), Bandi and Russell (2006), Barndorff-Nielsen *et al.* (2008). There are also connections with the growing literature on modeling daily volatility using intra-daily information. Research in this area includes Andersen *et al.* (2007), Patton and Sheppard (2009), Shephard and Sheppard (2010), Hansen *et al.* (2012) and Chen *et al.* (2011). Chiriac and Voev (2011) and Noureldin *et al.* (2012a) explore models for realized covariance matrices. This work also fits with the larger segment of the literature that finds evidence of long range dependence in volatility and have proposed ways to capture it. Significant contributions include long memory models (Andersen *et al.* (2003), Deo *et al.* (2006), Andersen *et al.* (2007), Corsi (2010), and Luciani and Veredas (2011)). Finally, this paper relates to the recent strand of literature on panel volatility modeling. Contributions in this area include Bauwens and Rombouts (2007), Engle *et al.* (2008), Engle (2009), Pakel *et al.* (2011), Wang and Zou (2010), Hautsch *et al.* (2011) and Noureldin *et al.* (2012b).

The paper is structured as follows. Section 2 describes the panels of realized volatility measures that we use in the empirical application and reports some descriptive statistics that motivate our modelling approach. Section 3 describes our specification and Section 4 details the estimation strategy. The asymptotic properties of the estimator are given in Section 5. In Section 6 we carry out a Monte Carlo exercise to assess the reliability of the proposed estimation approach. Section 7 presents the estimation results for the SPDR sectoral indices and the constituents of the S&P100. We conclude in Section 8. Assumptions, proofs and additional empirical results are gathered in Appendix A, B, and C, respectively.

2 Stylized Facts for Panels of Volatility Measures

We study two panels of realized volatility measures from January 2, 2001 to December 31, 2008. The first, referred to as SPDR, consists of the nine Select Sector SPDRs Exchange Traded Funds (ETF) that divide the S&P500 index into sector index funds. The sectors (with the abbreviations we use and the original ticker names) are Materials (Mat, XLB), Energy (Ener, XLE), Financial (Fin, XLF), Industrial (Ind, XLI), Technology (Tech, XLK), Consumer Staples (Stap, XLP), Utilities (Util, XLU), Health Care (Heal, XLV), and Consumer Discretionary (Disc, XLY). The second panel, named S&P100, consists of U.S. equity companies that are part of the S&P100 index. It contains all the constituents of the S&P100 index as of December 2008 that have been trading in the full sample period (90 in total). The complete list of S&P100 tickers, company names and industry sectors is reported in Appendix C.

Among the available estimators of the daily integrated volatility based on intraday returns, we adopt the realized kernels (Barndorff-Nielsen *et al.* (2008)).[1] They are a family of heteroskedastic and autocorrelation consistent volatility estimators, robust to various forms of market microstructure noise present in high frequency data. Our choice is motivated by the appealing theoretical properties of this family of estimators, as well as their good forecasting performance (e.g. for predicting Value at Risk, Brownlees and Gallo (2010)). Parallel analysis using alternative estimators (not reported in the paper) suggests that our results do not hinge on the specific measure of volatility chosen.

We compute optimal realized kernels following the procedure detailed in Barndorff-Nielsen *et al.* (2009). Our primary source of data are tick–by–tick intra-daily quotes from the TAQ database. Data are extracted and filtered using the methods described in Brownlees and Gallo (2006) and Barndorff-Nielsen *et al.* (2009). Let $r_{i\,t\,j}$ denote the 1–minute frequency returns (sampled in tick time) at minute j on day t for ticker i. The realized Parzen kernel estimator is defined as

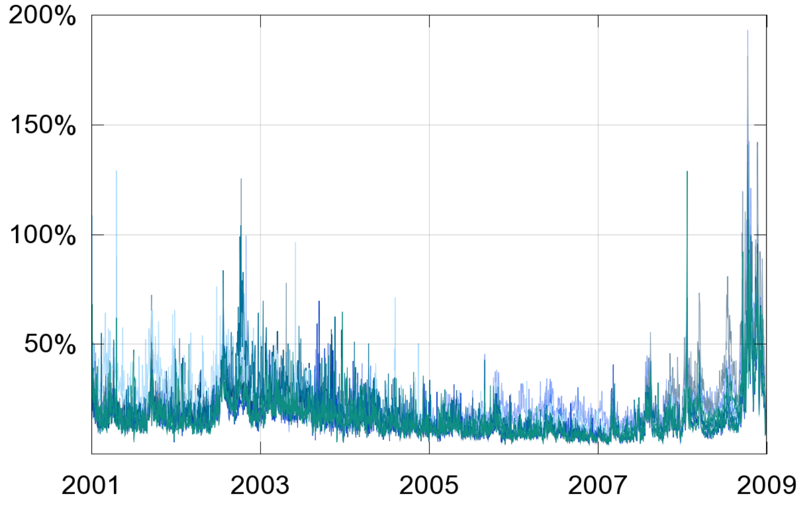
$$x_{i\,t} = \sum_{h=-H}^{H} \mathrm{K}_p\left(\frac{h}{H+1}\right)\gamma_h, \quad \text{with} \quad \gamma_h = \sum_{j=|h|+1}^{J} r_{i\,t\,j} r_{i\,t\,j-|h|},$$

and where H is both the bandwidth of the kernel and the maximum order of the autocovariance, J is the number of 1–minute frequency returns within the day, and $\mathrm{K}_p(\cdot)$ denotes the Parzen kernel. Under appropriate conditions, Barndorff-Nielsen *et al.* (2008) show that the realized kernel estimator converges to the integrated variance of returns. The computation of the estimator and optimal choice of the bandwidth parameter for each series closely follows the guidelines described by Barndorff-Nielsen *et al.* (2009).

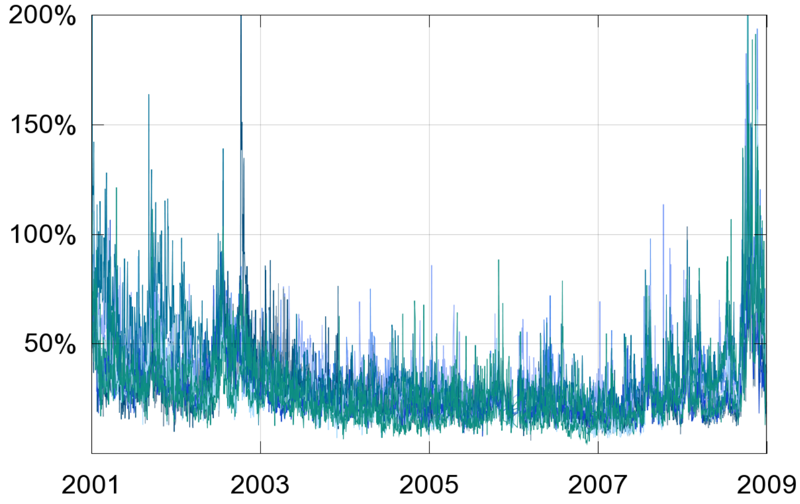
Figure 1 shows plots of the two panels of percent annualized volatility $\sqrt{252 x_{i\,t}}$; top for SPDR and bottom for S&P100. The plots suggest that the series cluster around a common time–varying average level that can be interpreted as *systematic volatility*. Statistical tests for the selection of the number of common factors in the panels strongly support the evidence of a one factor structure (see Luciani and Veredas (2011) for an exhaustive analysis and Andersen *et al.* (2001) for stylized facts on a similar panel). The secular movements of systematic volatility can be attached to well known economic events or system wide innovations. The high level of volatility in the beginning of the 2000s is related to the aftermath of dot–com bubble burst and the recession. The period that goes from 2004 to July 2007 is characterized by a low level of uncertainty that corresponds to the market rally following the recession. Finally, volatility rises with the advent of the financial crisis

[1] Other alternative estimators are the range (Parkinson (1980), Alizadeh *et al.* (2002)), the "vanilla" 5–minute realized volatility (Andersen *et al.* (2003)), or the two-scales estimator (Aït-Sahalia *et al.* (2005)).

Figure 1: Realized volatilities



(a) SPDR



(b) S&P100

Annualized realized volatilities for the SPDR select sectors (top) and the 90 S&P100 constituents (bottom) from January 2, 2001 to December 31, 2008.

and it skyrockets to the highest level reached over the last 20 years in the fall of 2008, following the demise of Lehman Brothers.

Table 1 displays descriptive statistics. The table reports average percent annualized volatility, standard deviation of volatility (volatility of volatility), daily, weekly (5 days) and monthly (22 days) autocorrelations, average correlation at lag 0 with the other series, and the percentage of variance explained by the first principal component. For SPDR we report statistics for each sectoral index while for S&P100 we report the 25%, 50%, and 75% quantiles of the statistics across industry sectors. Mean and variability levels of the series are higher for the S&P100 panel rather then the SPDR, due to the fact the sectoral aggregation decreases the average and dispersion of volatility. Autocorrelations decay slowly, consistently with the evidence of long range dependence widely documented in volatility studies. The average cross-correlation with the other series and the proportion of variance explained by the first principal components are always above 0.50 and 50% respectively, which confirms the existence of strong co–movements in volatility.

Table 1: Descriptive statistics

		vol	vov	$\hat{\rho}_{\mathsf{day}}$	$\hat{\rho}_{\mathsf{week}}$	$\hat{\rho}_{\mathsf{month}}$	$\bar{\rho}$	PC_1
				SPDR				
Mat		23.34	11.83	0.72	0.63	0.39	0.82	0.91
Ener		25.68	12.78	0.65	0.61	0.35	0.79	0.87
Fin		25.85	15.71	0.69	0.50	0.35	0.77	0.84
Ind		21.85	11.29	0.66	0.57	0.37	0.81	0.88
Tech		26.82	13.34	0.49	0.41	0.27	0.66	0.58
Stap		16.93	8.21	0.45	0.36	0.19	0.77	0.77
Util		24.12	12.73	0.65	0.55	0.35	0.72	0.70
Heal		17.27	8.61	0.34	0.27	0.17	0.73	0.68
Disc		20.56	10.57	0.64	0.55	0.35	0.82	0.90
				S&P100				
Mat	$q_{0.25}$	33.80	14.49	0.66	0.55	0.31	0.63	0.62
	$q_{0.50}$	34.73	15.17	0.69	0.56	0.34	0.67	0.72
	$q_{0.75}$	36.42	16.59	0.73	0.60	0.37	0.72	0.83
Ener	$q_{0.25}$	34.00	15.04	0.57	0.50	0.26	0.61	0.61
	$q_{0.50}$	37.98	16.36	0.66	0.57	0.32	0.68	0.72
	$q_{0.75}$	44.28	18.50	0.71	0.66	0.38	0.69	0.74
Fin	$q_{0.25}$	36.45	21.01	0.63	0.27	0.17	0.52	0.49
	$q_{0.50}$	39.77	23.77	0.66	0.47	0.28	0.64	0.70
	$q_{0.75}$	42.93	26.59	0.74	0.56	0.34	0.68	0.76
Ind	$q_{0.25}$	29.25	12.42	0.61	0.49	0.31	0.65	0.64
	$q_{0.50}$	30.61	13.44	0.64	0.54	0.33	0.69	0.73
	$q_{0.75}$	33.19	17.33	0.70	0.56	0.36	0.71	0.80
Tech	$q_{0.25}$	34.96	16.10	0.64	0.52	0.34	0.56	0.49
	$q_{0.50}$	38.53	17.85	0.67	0.58	0.37	0.62	0.58
	$q_{0.75}$	46.20	23.26	0.72	0.61	0.40	0.72	0.78
Util	$q_{0.25}$	28.24	13.58	0.67	0.45	0.25	0.60	0.54
	$q_{0.50}$	29.61	14.09	0.70	0.52	0.30	0.67	0.66
	$q_{0.75}$	31.42	15.20	0.72	0.60	0.34	0.70	0.73
Stap	$q_{0.25}$	25.46	11.43	0.46	0.36	0.22	0.60	0.52
	$q_{0.50}$	28.32	12.45	0.53	0.43	0.24	0.68	0.69
	$q_{0.75}$	29.80	13.30	0.58	0.47	0.30	0.71	0.76
Heal	$q_{0.25}$	29.19	13.02	0.48	0.34	0.22	0.64	0.62
	$q_{0.50}$	30.06	13.55	0.59	0.46	0.29	0.65	0.63
	$q_{0.75}$	32.95	15.00	0.61	0.50	0.34	0.66	0.67
Disc	$q_{0.25}$	32.98	15.49	0.58	0.47	0.30	0.63	0.60
	$q_{0.50}$	35.76	17.41	0.64	0.54	0.36	0.68	0.68
	$q_{0.75}$	41.13	21.01	0.67	0.58	0.41	0.70	0.76

The top part shows descriptive statistics for SPDR. The bottom part shows the same descriptive statistics for S&P100 with the assets grouped according to the same sectors as SPDR. For each group the table shows the 25, 50 and 75 quantiles. The columns report the average annualized volatility (vol), standard deviation of volatility, volatility of volatility (vov), the autocorrelations of order 1, 5 and 22 ($\hat{\rho}_{\mathsf{day}}$, $\hat{\rho}_{\mathsf{week}}$ and $\hat{\rho}_{\mathsf{month}}$), the average cross–correlation with the other series in the dataset ($\bar{\rho}$), and the percentage of the variance explained by the first principal component (PC_1).

3 A Seminonparametric Vector MEM

The empirical evidence of the previous section suggests that the dynamics of the volatility measures in the panel can be described by a common secular trend and residual idiosyncratic short run components. In this section we introduce a novel Seminonparametric Vector Multiplicative Error Model (SPvMEM) that captures these empirical regularities.

Let $x_{i\,t}$ be the value of the realized volatility measure for the i^{th} asset in period t, with $i = 1, ..., N$ and $t = 1, ..., T$, and let $z_t = t/T$ denote the (rescaled) time index. The realized measure $x_{i\,t}$ is modelled in a multiplicative specification of the form

$$x_{i\,t} = a_i\,\phi(z_t)\,\mu_{i\,t}\,\epsilon_{i\,t} \quad \epsilon_{i\,t}|\mathcal{F}_{t-1} \sim D(1), \tag{1}$$

where $\mathcal{F}_{t-1}$ is the information set up to time $t-1$, a_i is the scale factor of the i^{th} series, $\phi(z_t)$ is a deterministic time trend, $\mu_{i\,t}$ is an idiosyncratic short term dynamic component and $\epsilon_{i\,t}$ is a conditionally independent error term with positive support, unit expectation and independent of z_t. More detailed assumptions on the process are given in Appendix A.

The component $\phi(z_t)$ is a scalar smooth function capturing the low frequency common trend. It is assumed that $\phi : [0, 1] \to \mathcal{P} \subset \mathbb{R}_+$ and that ϕ belongs to the set $\Gamma = \{p \in C^\infty[0, 1] : p(z_t) \in \mathcal{P}$ for all $z_t \in [0, 1]\}$. The use of $z_t = t/T$ as a regressor is a common assumption in models with time–varying parameters. In order to derive the asymptotic results, it is useful to think of t/T as a draw from a uniform distribution on $[0, 1]$. The trend is further assumed to have unit mean, that is $\mathrm{E}[\phi(z_t)] = 1$. In what follows, we denote ϕ without any reference to z_t as an infinite dimensional nuisance parameter belonging to Γ in the sense of Severini and Wong (1992). Our specification choice of the trend has important implications for estimation. In a rescaled time framework, i.e. when $z_t = t/T$, as T increases the number of observations in a neighborhood of each point of the trend increases as well, and this allows to carry out pointwise inference on the trend. Also, note that conditionally on z_t the process is stationary.

The idiosyncratic component $\mu_{i\,t}$ is a nonnegative conditionally predictable scalar process with unit mean. It is defined as

$$\mu_{i\,t} = \mu_i(\mathcal{F}_{t-1}, \boldsymbol{\delta}_i)$$

where $\boldsymbol{\delta}_i \in \mathcal{D}_i \subset \mathbb{R}^{p_\delta}$, is a vector of parameters characterizing the dynamics of the process. Several functional forms for $\mu_{i\,t}$ have been proposed in the MEM literature. Here, we opt for asymmetric GARCH type dynamics, that is

$$\mu_{i\,t} = \left(1 - \alpha_i - \beta_i - \frac{\gamma_i}{2}\right) + \alpha_i \frac{x_{i\,t-1}}{a_i \phi(z_{t-1})} + \beta_i \mu_{i\,t-1} + \gamma_i \frac{x_{i\,t-1}}{a_i \phi(z_{t-1})} \mathbf{1}_{\{r_{i\,t-1} < 0\}}, \tag{2}$$

where $\alpha_i > 0$, $\gamma_i \geq 0$ and $\beta_i \geq 0$, and $r_{i\,t-1}$ denotes the return on day $t-1$ of asset i. Thus, in this case $\boldsymbol{\delta}_i = (\alpha_i, \beta_i, \gamma_i)^\intercal$ and $p_\delta = 3$. This type of functional form is often used in the literature (see Engle and Rangel, 2008) and it is typically found to be successful for prediction. We assume that that the probability of $\{r_{i\,t-1} < 0\}$ is 1/2 and that $\mu_{i\,t}$ is stationary, that is $\alpha_i + \gamma_i/2 + \beta_i < 1$. Under these assumptions it is straightforward to check that the unconditional mean of (2) is one. Idiosyncratic dynamics can be equivalently parametrized in an alternative way. Let $\tilde{\mu}_{i\,t} = a_i \mu_{i\,t}$, then

$$\tilde{\mu}_{i\,t} = \omega_i + \alpha_i \frac{x_{i\,t-1}}{\phi(z_{t-1})} + \beta_i \tilde{\mu}_{i\,t-1} + \gamma_i \frac{x_{i\,t-1}}{\phi(z_{t-1})} \mathbf{1}_{\{r_{i\,t-1} < 0\}},$$

with $\omega_i = a_i \left(1 - \alpha_i - \beta_i - \frac{\gamma_i}{2}\right)$. This alternative parameterization is convenient for estimation, as we will see later.

The conditional moments of the process are

$$\mathrm{E}[x_{i\,t}|\mathcal{F}_{t-1}] = a_i\phi(z_t)\mu_{i\,t} \quad \text{and} \quad \mathrm{Var}[x_{i\,t}|\mathcal{F}_{t-1}] = a_i^2\phi^2(z_t)\mu_{i\,t}^2\mathrm{Var}(\epsilon_{i\,t}).$$

Thus, the SPvMEM is a conditionally heteroskedastic process where the conditional mean and variance change over time and are driven by the level of the common trend and of the idiosyncratic dynamics. On the other hand, the trend adjusted process $x_{i\,t}/\phi(z_t)$ is standard stationary MEM.

The model can be compactly expressed using vector notation. Let $\mathbf{x}_t = (x_{1\,t}, \ldots, x_{N\,t})^{\mathrm{T}}$ be the $N \times 1$ dimensional vector of volatility measures. Let $\mathbf{a} = (a_1, \ldots, a_N)^{\mathrm{T}} \in \mathcal{A} \subset \mathbb{R}_+^N$ and let the idiosyncratic dynamics be represented as a $N \times 1$ vector $\boldsymbol{\mu}(\mathcal{F}_{t-1}, \boldsymbol{\delta}) \subset \mathbb{R}_+^N$:

$$\boldsymbol{\mu}(\mathcal{F}_{t-1}, \boldsymbol{\delta}) = \boldsymbol{\mu}_t = \begin{pmatrix} \mu_1(\mathcal{F}_{t-1}, \boldsymbol{\delta}_1) \\ \mu_2(\mathcal{F}_{t-1}, \boldsymbol{\delta}_2) \\ \vdots \\ \mu_N(\mathcal{F}_{t-1}, \boldsymbol{\delta}_N) \end{pmatrix},$$

with $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_N) \in \mathcal{D} \subset \mathbb{R}^{Np_\delta}$. The SPvMEM can be written as

$$\mathbf{x}_t = \phi(z_t) \cdot \mathbf{a} \odot \boldsymbol{\mu}_t \odot \boldsymbol{\epsilon}_t, \tag{3}$$

where $\boldsymbol{\epsilon}_t = (\epsilon_{1\,t}, ..., \epsilon_{N\,t})'$ and $\odot$ denotes the Hadamard component–wise product operator.

We complete the definition of the model described in Equation (3) with the specification of the distribution of the error term $\boldsymbol{\epsilon}_t$ which is a $N \times 1$ vector of innovations. Its probability density function (pdf) and cumulative density function (cdf) are denoted by $f_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}_t; \boldsymbol{\theta})$ and $F_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}_t; \boldsymbol{\theta})$ respectively, where $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subset \mathbb{R}^{p_\theta}$.

Choosing an appropriate multivariate specification for the innovation term can be challenging in this setting as there are few multivariate distributions for positive real–valued random vectors (see Johnson *et al.* (2000)). The multivariate Exponential and Gamma are the two most prominent but they are cumbersome and their properties may not always dovetail with those of the volatility measures. Building up on Cipollini *et al.* (2006), we follow a modelling strategy based on copulas. Let $F_{\epsilon_i}(\epsilon_{i\,t})$ denote the marginal cdf of $\epsilon_{i\,t}$ and define $u_{i\,t} = F_{\epsilon_i}(\epsilon_{i\,t})$ for $i = 1, \ldots, N$, then, by Sklar's theorem the joint cdf and pdf of $\boldsymbol{\epsilon}_t$ can be written respectively as

$$F_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}_t) = C(u_{1\,t}, \ldots, u_{N\,t}) \quad \text{and} \quad f_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}_t) = \prod_{i=1}^{N} f_{\epsilon_i}(\epsilon_{i\,t}) \cdot c(u_{1\,t}, \ldots, u_{N\,t}), \tag{4}$$

where $C(\cdot)$ is the copula function and $c(\cdot)$ is its derivative with respect to $(u_{1\,t}, \ldots, u_{N\,t})$. The advantage of copulas is that they allow us to decompose the task of specifying the distribution for $\boldsymbol{\epsilon}_t$ in two subtasks: the choice of the marginal pdfs $f_{\epsilon_i}(\cdot)$ and the copula density $c(\cdot)$.

As far as the choice of the marginal distribution is concerned, we opt for a Gamma distribution. It is a flexible distribution, it belongs to the exponential family, and it nests the Exponential, Chi–Square, Erlang, and Maxwell–Boltzmann distributions. If the marginal distribution of $\epsilon_{i\,t}$ is a Gamma distribution with parameters (k_i, ν_i), then the marginal conditional distribution of $x_{i\,t}$ is also Gamma

$$x_{i\,t}|\mathcal{F}_{t-1} \sim Gamma\left((a_i\phi(z_t)\mu_{i\,t})^{-1}k_i, \nu_i \right)$$

with conditional pdf

$$f_{x_i}(x_{i\,t};|\mathcal{F}_{t-1}) = \frac{k_i}{\Gamma(\nu_i)a_i\phi(z_t)\mu_{i\,t}}\left(\frac{x_{i\,t}k_i}{a_i\phi(z_t)\mu_{i\,t}}\right)^{\nu_i-1}\exp\left(-\frac{x_{i\,t}k_i}{a_i\phi(z_t)\mu_{i\,t}}\right). \tag{5}$$

In what follows, we fix $k_i = \nu_i$ to ensure that $\epsilon_{i\,t}$ has unit mean, while $\mathrm{Var}[\epsilon_{i\,t}] = \nu_i^{-1}$.

As far as the choice of the copula function is concerned, we adopt a Gaussian meta–copula, as delivers estimators that are easy to compute numerically in large dimensions:

$$c_\Phi(u_{1\,t},\ldots,u_{N\,t};\mathbf{R}) = |\mathbf{R}|^{-1/2}\exp\left(-\frac{1}{2}\left(\Phi^{-1}(\mathbf{u}_t)\right)^{\mathrm{T}}(\mathbf{R}-\mathbf{I})\left(\Phi^{-1}(\mathbf{u}_t)\right)\right) \tag{6}$$

where $\Phi(\cdot)$ is the Gaussian cdf, $\mathbf{R}$ is the correlation matrix and $\mathbf{I}$ is the identity matrix. The marginal and copula parameters are collected in the vector $\boldsymbol{\theta}$ defined as $(\nu_1\ldots\nu_N,\mathrm{vech}(\mathbf{R}))^{\mathrm{T}}$, which has a dimension of $N + N(N-1)/2$.

Our choices of the marginal and copula function are primarily driven on the grounds of simplicity (Song (2000), Cipollini *et al.* (2006)). We acknowledge that the Gamma and Gaussian meta–copula have some limitations in fitting the moments of the data in empirical applications, and that it might be of interest adopting more sophisticated distributions. For instance, in some applications measuring tail dependences may be of interest, such as in financial derivatives based on volatilities.

Nevertheless, the theory developed in this paper carries through different choices of the marginal densities and for any copula function. As for the marginals, any distribution belonging to the exponential family delivers simple closed form estimators for ϕ. Moreover, by Quasi Maximum Likelihood arguments (see Engle and Gallo, 2006), if the conditional moments are correctly specified the marginals deliver consistent estimates of the parameters. Concerning the copula choice, the theory developed in this work allows for general types of copulas, and inference under possible copula misspecification can be addressed by adapting Chen and Fan (2006) to our theory.

4 Estimation Procedure

The specification introduced in the previous section is a nonlinear model containing both parametric and nonparametric elements. Since joint estimation of both components is cumbersome, we propose a three step estimation procedure that stems from profile likelihood estimation (Severini and Wong, 1992) and inference from the marginals (Joe, 1997, 2005): i) nonparametric estimation of the common trend ϕ; ii) estimation of the marginal parameters $\boldsymbol{\xi}_i = (a_i, \boldsymbol{\delta}_i^{\mathrm{T}}, \nu_i)^{\mathrm{T}}$ of each series; and iii) estimation of the copula parameter $\boldsymbol{\psi} = \mathrm{vech}(\mathbf{R})$.

i) Conditionally on the idiosyncratic dynamics $\mu_{i\,t}$ and the marginal parameters $\boldsymbol{\xi}_i$, a natural estimator of the common trend is a Nadaraya–Watson type estimator applied to the weighted average of the rescaled series. That is, for any $z_\tau \in [0,1]$,

$$\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) = \frac{\sum_{t=1}^{T}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\sum_{i=1}^{N}\frac{x_{i\,t}}{a_i\mu_{i\,t}}\frac{\nu_i}{\sum_{i=1}^{N}\nu_i}}{\sum_{t=1}^{T}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)}, \tag{7}$$

where $\mathrm{K}(\cdot)$ is a suitable kernel function with a bandwidth h_{NT} that can vary both with N and T (see the next section for details). This estimator has an intuitive meaning: the common trend at

Table 2: SPvMEM Estimation Algorithm

Iteration 0

0.1 For each $t \in [0, T]$ define $z_t = t/T$ and compute the initial estimate:

$$\widehat{\phi}^0(z_\tau) = \frac{\sum_{t=1}^{T} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \sum_{i=1}^{N} \frac{x_{i\,t}}{\bar{x}_i} \frac{1/s_i^2}{\sum_{i=1}^{N} 1/s_i^2}}{\sum_{t=1}^{T} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)},$$

where $\bar{x}_i$ and s_i^2 are, respectively, sample mean and variance of the i^{th} series.

0.2 For each $i = 1, \ldots, N$ set $(\boldsymbol{\xi}_1^0, \ldots, \boldsymbol{\xi}_N^0)$ where $\boldsymbol{\xi}_i^0 = (a_i^0, \alpha_i^0, \beta_i^0, \gamma_i^0, \nu_i^0)$

Iteration $q > 0$

1 For each $i = 1, \ldots, N$ maximize the N log–likelihoods

$$\widehat{\boldsymbol{\xi}}_i^q = \arg\max_{\boldsymbol{\xi}_i} \sum_{t=1}^{T} \log f_{x_i}(x_{i\,t}; \boldsymbol{\xi}_i, \widehat{\phi}^{q-1} | \mathcal{F}_{t-1})$$

2 Compute the N idiosyncratic components evaluated at $\widehat{\phi}^{q-1}$

$$\widehat{\mu}_{i\,t}^q = \left(1 - \widehat{\alpha}_i^q - \widehat{\beta}_i^q - \frac{\widehat{\gamma}_i^q}{2}\right) + \widehat{\alpha}_i^q \frac{x_{i\,t-1}}{\widehat{a}_i^q \widehat{\phi}^{q-1}(z_{t-1})} + \widehat{\beta}_i^q \hat{\mu}_{i\,t-1}^q + \widehat{\gamma}_i^q \frac{x_{i\,t-1}}{\widehat{a}_i^q \widehat{\phi}^{q-1}(z_{t-1})} \mathbf{1}_{r_{i\,t-1}<0}.$$

For each $z_t \in [0, 1]$, given $\hat{\mu}_{i\,t}^q$, compute:

$$\widehat{\phi}^q(z_t) = \frac{\sum_{t=1}^{T} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \sum_{i=1}^{N} \frac{\widehat{\nu}_i}{\sum_{i=1}^{N} \widehat{\nu}_i} \frac{x_{i\,t}}{\widehat{a}_i^q \widehat{\mu}_{i\,t}^q}}{\sum_{t=1}^{T} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)}.$$

3 Check for convergence otherwise go back to 1

Copula Estimation

1 The Gaussian meta-copula is estimated with the sample covariance matrix of $\Phi^{-1}(\hat{u}_{i\,t})$ where $\hat{u}_{i\,t} = \widehat{F}_{\epsilon_i}(\hat{\epsilon}_{i\,t})$

z_τ is estimated as a nonparametric regression of a weighted sum (across N) of $x_{i\,t}$ adjusted by the idiosyncratic component $a_i \mu_{i\,t}$, where the weights are $\nu_i / \sum_{i=1}^{N} \nu_i$. Since ν_i is the reciprocal of the variance of the i^{th} innovation, the weights also have an intuitive interpretation: the least erratic series $x_{i\,t}$ (denoted by larger ν's) receive more weight in the estimation of the common trend. The technical justification for this weighting scheme is given in the next section when the estimator is derived. Other weighting schemes are also possible, e.g. equal weights $1/N$. As far as the choice of the kernel function is concerned, it is important to stress that such a choice ought to be based on the purpose of the application of the SPvMEM. If the objective of the application is in–sample estimation then a two–sided kernel is more suitable, and the next section develops the related asymptotic properties of the estimated curve in this setting. On the other hand, if the objective of the application is out–of–sample forecasting, then one should resort to a one–sided kernel (see Gijbels *et al.*, 1999).

ii) Analogously, conditionally on the common component ϕ, we can adjust each series for the level of the common trend. It follows from the properties of the Gamma distribution that

$$\left.\frac{x_{it}}{\phi(z_t)}\right| \mathcal{F}_{t-1} \sim Gamma\left((a_i\mu_{it})^{-1}\nu_i, \nu_i\right). \tag{8}$$

Thus, given an estimator of the common trend, we can estimate the marginal parameter vector $\boldsymbol{\xi}_i = (a_i, \boldsymbol{\delta}_i^{\intercal}, \nu_i)^{\intercal}$ by maximising the marginal likelihood of the trend adjusted series associated to (8) for each series in the panel. In this way we obtain the estimator $\widehat{\boldsymbol{\xi}}_{iT}$.

iii) Finally, conditionally on the first two steps, the correlation $\mathbf{R}$ across the innovations can be estimated as the sample covariance matrix of the transformed marginal cdfs $\Phi^{-1}(u_{it}) = \Phi^{-1}(F_{x_i}(x_{it}; \boldsymbol{\xi}_i, \phi|\mathcal{F}_{t-1}))$.

These considerations motivate us to propose an iterative procedure to estimate the SPvMEM. The estimation strategy we propose consists of iteratively estimating the first and second step until convergence and then estimating the copula parameters. The explanation of the algorithmic procedure is detailed in Table 2. The procedure is initialized as follows. The initial estimate of the trend requires an educated guess of $a_i\mu_{it}$ and ν_i, for all i (step 0.1). As for $a_i\mu_{it}$, since it is the conditional mean of the i^{th} risk adjusted for the common component, we consider the sample means $\bar{x}_i$. As for ν_i, since it is the inverse of the variance of a Gamma distribution, we consider the inverse of the sample variance s_i^2 (see Hafner and Linton, 2010). The values of the marginal parameters are also set to an educated initial guess (step 0.2). The parameters α_i^0, β_i^0 and γ_i^0 are set respectively 0.1, 0.8 and 0.1, while a_i is set to $(1 - \alpha_i^0 - \beta_i^0 - \gamma_i^0)\bar{x}_i$, and ν_i to $1/s_i^2$. After the initial parameters are initialized, the estimation algorithm proceeds as follows. The estimated parameters at iteration q are obtained by maximizing the N log–likelihoods using an estimate of the curve obtained at iteration $q-1$ (step 1). Next, using the updated estimates of the conditional dynamics μ_{it}, we update the common trend estimation (step 2). Steps 1 and 2 are repeated until convergence of the estimated parameters. Convergence is typically achieved within a few iterations. Finally, the Gaussian meta–copula is estimated with the sample covariance matrix of the transformed uniformed residuals $\Phi^{-1}(\hat{u}_{it})$ where $\hat{u}_{it} = \widehat{F}_{\epsilon_i}(\hat{\epsilon}_{it})$.

5 Asymptotic Theory

This section establishes the asymptotic properties of our estimators. We establish consistency and asymptotic Gaussianity of the common trend and of the parameters. The asymptotic framework considered in this section is developed for $T \to \infty$ and both N fixed and $N \to \infty$ and all the assumptions and proofs are in Appendix A and B respectively. An interesting feature of our setting is that consistent estimation of the common trend does not necessarily require a large cross–section. As the sample size T increases, the number of observations in the neighborhood of each point of the trend increases as well and this allows for consistent inference. However, we show in this section that, when also the cross–sectional dimension N is allowed to increase to infinity, consistency is still achieved and some of the inferential procedures simplify. This result is obtained if we add two additional conditions, one on the choice of the kernel bandwidth and one on the dependence captured by the copula. In particular, while the former condition allows for a smaller bandwidth, the latter condition requires a weaker dependence explained by the copula. It is the analog of the condition imposed on the dependence between idiosyncratic components in linear models (see e.g. Bai and Ng (2002)).

We adopt the following notation for the log–likelihood of the SPvMEM. The model contains an infinite dimensional nuisance parameter ϕ and a $5N + N(N-1)/2$–dimensional vector of parameters $\boldsymbol{\eta} = (\mathbf{a}^{\mathrm{T}}, \boldsymbol{\delta}^{\mathrm{T}}, \boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}$. The log–likelihood of observation $\mathbf{x}_t$ at time t conditional on the information set $\mathcal{F}_{t-1}$ is defined as

$$\boldsymbol{\ell}_t(\boldsymbol{\eta}, \phi) = \log f_{\mathbf{x}}(\mathbf{x}_t; \boldsymbol{\eta}, \phi|\mathcal{F}_{t-1}), \tag{9}$$

using the notation $\boldsymbol{\ell}_t(\boldsymbol{\eta}, \phi(z_t))$ when necessary. It is convenient to rearrange the parameter vector $\boldsymbol{\eta}$. We collect the parameters of the marginals in a $5N$–dimensional vector $\boldsymbol{\xi} = (\boldsymbol{\xi}_1^{\mathrm{T}} \dots \boldsymbol{\xi}_N^{\mathrm{T}})^{\mathrm{T}}$, where $\boldsymbol{\xi}_i = (a_i, \boldsymbol{\delta}_i^{\mathrm{T}}, \nu_i)^{\mathrm{T}} \in \Xi_i \subset \mathbb{R}^{5N}$ and such that $\Xi_i \cap \Xi_j = \emptyset$ for $i \neq j$. The parameters of the copula are collected in a vector $\boldsymbol{\psi} \in \boldsymbol{\Psi} \subset \mathbb{R}^{p_\psi}$. It follows from the joint pdf of the innovations in equation (4) and Sklar's theorem that the conditional log–likelihood can be expressed as

$$\mathcal{L}_{NT}(\boldsymbol{\eta}, \phi) = \sum_{t=1}^{T} \sum_{i=1}^{N} \boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_i, \phi) + \sum_{t=1}^{T} \ell_t^{c}(\boldsymbol{\xi}, \boldsymbol{\psi}, \phi), \tag{10}$$

where

$$\begin{aligned}
\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_i, \phi) &= \log f_{x_i}(x_{it}; \boldsymbol{\xi}_i, \phi|\mathcal{F}_{t-1}), \quad \text{and} \\
\boldsymbol{\ell}_t^{c}(\boldsymbol{\xi}, \boldsymbol{\psi}, \phi) &= \log c(u_{1t}, \dots, u_{Nt}; \boldsymbol{\psi}|\mathcal{F}_{t-1}).
\end{aligned}$$

Again, we also use the notation $\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_i, \phi(z_t))$ and $\boldsymbol{\ell}_t^{c}(\boldsymbol{\xi}, \boldsymbol{\psi}, \phi(z_t))$ when necessary.

In the sequel we use a more detailed notation. We denote by $\boldsymbol{\eta}_0 = (\boldsymbol{\xi}_{1\,0}, \dots, \boldsymbol{\xi}_{N\,0}, \boldsymbol{\psi}_0)$ the true values of the parameters while by ϕ_0 we indicate the true curve and we denote by E_0 the expectation taken under the true model.

In an $i.i.d.$ univariate context and based on Staniswalis (1987, 1989), Severini and Wong (1992) propose and prove the asymptotic properties of an estimator of ϕ_0 based on smoothed profile log–likelihood maximization. This estimator has been generalized to the univariate dependent case by Veredas *et al.* (2007). In the multivariate context of the SPvMEM this technique requires an estimator of $\boldsymbol{\eta}_0$, which in principle we could obtain by maximizing the joint log–likelihood (9). The procedure to estimate ϕ_0 and $\boldsymbol{\eta}_0$ would be based on iterating between two optimizations: on one hand the global and parametric optimization with respect to the $5N$ parameters in $\boldsymbol{\xi}_0 = (\boldsymbol{\xi}_{1\,0}, \dots, \boldsymbol{\xi}_{N\,0})$ plus the p_ψ parameters in $\boldsymbol{\psi}_0$ and, on the other hand and at each iteration, the optimization of the localized (or smoothed) log–likelihood has to be performed T times. This procedure has two main drawbacks. First, it is computationally intensive, and, second, it requires an expression for the joint log–likelihood. Even for small values of N this approach seems to be unfeasible and can be simplified by means of inference from the marginals, which leads to the estimator introduced in the previous section.

We first show how to derive the estimator of the common trend ϕ_0, followed by the analysis of the effect of the nonparametric component on the inference on $\boldsymbol{\xi}_0$. In Theorems 1–2 we show the asymptotic properties of $\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}$, defined in (7), under the true parameters of the marginals $\boldsymbol{\xi}_0$. In Theorem 3 and Corollary 1 we study the relation between the estimator of the curve and a generic value of the parameters $\boldsymbol{\xi}$. In Theorem 4 we show consistency and the asymptotic distribution of $\widehat{\boldsymbol{\xi}}_T$. Finally, in Theorem 5 we prove consistency of $\widehat{\boldsymbol{\psi}}_T$. Theorems 1–4 are proved in the case of fixed N and large T, and both N and T large.

5.1 Estimation of the common trend

Given the marginal likelihoods, for any $z_\tau \in [0,1]$ and for a given value of the parameters $\boldsymbol{\xi}$, then, the estimator of the curve is such that

$$\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) = \arg\sup_{\phi\in\Gamma} \sum_{t=1}^{T} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \sum_{i=1}^{N} \boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_i, \phi(z_t)). \tag{11}$$

Here $\mathrm{K}(\cdot)$ is a suitable kernel function satisfying assumption K in Appendix A and h_{NT} is a bandwidth parameter that can depend on both N and T, as explained in Theorem 2 below. Given the choice of Gamma distributions for the marginals, this optimization has a closed form solution

$$\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) = \frac{\sum_{t=1}^{T} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \sum_{i=1}^{N} \frac{x_{it}}{a_i \mu_{it}} \frac{\nu_i}{\sum_{i=1}^{N}\nu_i}}{\sum_{t=1}^{T} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)}. \tag{12}$$

This is the estimator proposed in (7) and this derivation shows that the weighting scheme based on the innovation variances is actually a direct consequence of the maximization in (11).

The following Theorems show consistency and asymptotic Gaussianity of $\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau)$, i.e. of the estimator in (12) when $\boldsymbol{\xi} = \boldsymbol{\xi}_0$.

Theorem 1 – Nonparametric component – Consistency *Given the estimator $\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau)$ and under assumptions A, B, C.1, I, K, L in Appendix A, and if $NTh_{NT} \to \infty$ and $h_{NT} \to 0$ as $NT \to \infty$ we have:*

a) if $T \to \infty$ and N is finite, then, for any $z_\tau \in [0,1]$,

$$\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau) \xrightarrow{P} \phi_0(z_\tau);$$

b) if $T \to \infty$ and $N \to \infty$ and assumption C.2 in Appendix A holds, the result in part a) still holds.

Theorem 2 – Nonparametric component – Asymptotic Gaussianity *Under assumptions A, B, C.1, I, K, L, P in Appendix A, and if $NTh_{NT} \to \infty$ as $NT \to \infty$ and one of the following conditions holds true:*

a) $T \to \infty$, N finite, and $h_{NT} \to 0$;

b) $T \to \infty$, $N \to \infty$, $Nh_{NT}^2 \to 0$, and assumptions C.2 and D in Appendix A hold;

then, the estimator $\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau)$ in (12) is such that

$$\sqrt{NTh_{NT}}\left(\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau) - \phi_0(z_\tau)\right) \xrightarrow{d} \mathcal{N}\left(0, V_{\boldsymbol{\xi}_o}(z_\tau)\right),$$

where, in case a)

$$
\begin{aligned}
V_{\boldsymbol{\xi}_o}(z_\tau) &= \left(\int_{-1}^{1}\mathrm{K}^2(u)\mathrm{d}u\right)\frac{\bar{\imath}_{N\,\boldsymbol{\xi}_o}(z_\tau)}{\bar{\jmath}^2_{N\,\boldsymbol{\xi}_o}(z_\tau)},\\
\bar{\imath}_{N\,\boldsymbol{\xi}_o}(z_\tau) &= \frac{1}{N}\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_{i\,o},\phi_0(z_\tau))\right)^2\right],\\
\bar{\jmath}_{N\,\boldsymbol{\xi}_o}(z_\tau) &= -\frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_{i\,o},\phi_0(z_\tau))\right],
\end{aligned}
$$

and in case b)

$$
\begin{aligned}
V_{\boldsymbol{\xi}_o}(z_\tau) &= \left(\int_{-1}^{1}\mathrm{K}^2(u)\mathrm{d}u\right)\frac{i_{\boldsymbol{\xi}_o}(z_\tau)}{j^2_{\boldsymbol{\xi}_o}(z_\tau)},\\
i_{\boldsymbol{\xi}_o}(z_\tau) &= \lim_{N\to\infty}\bar{\imath}_{N\,\boldsymbol{\xi}_o}(z_\tau),\quad j_{\boldsymbol{\xi}_o}(z_\tau) = \lim_{N\to\infty}\bar{\jmath}_{N\,\boldsymbol{\xi}_o}(z_\tau).
\end{aligned}
$$

Several remarks on these Theorems are in order. First, the estimator $\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}$ converges to the true curve ϕ_0 by virtue of Lemma 1 in Appendix B, which proves that ϕ_0 is a maximizer not only of the localized version of the joint log–likelihood (10) but also of each localized marginal log–likelihood. Second, with respect to Veredas *et al.* (2007), asymptotic efficiency is lost since we are neglecting the copula part of the log–likelihood (10). As for the effect of large N, we emphasize two results. In the limit $N,T\to\infty$, the bandwidth has to decrease at a faster rate, i.e. the local averages have to be computed using a smaller window of observations when the cross–section is large in comparison to when it is finite. Moreover the term $i_{\boldsymbol{\xi}_o}(z_\tau)$ in the asymptotic variance contains not only the variances of the marginal scores but also their covariances. Thus the asymptotic variance is well defined also in the limit $N\to\infty$, provided that we assume these covariances to be bounded (see assumption D). In particular, we have[2]

$$
\begin{aligned}
\bar{\imath}_{N\,\boldsymbol{\xi}_o}(z_\tau) &= \frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N}\left(\frac{\partial}{\partial\phi}\boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_{i\,o},\phi_0(z_\tau))\right)^2\right]+\\
&\quad+\frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left(\frac{\partial}{\partial\phi}\boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_{i\,o},\phi_0(z_\tau))\right)\left(\frac{\partial}{\partial\phi}\boldsymbol{\ell}^m_{j\,t}(\boldsymbol{\xi}_{j\,o},\phi_0(z_\tau))\right)\right]=\\
&= \frac{1}{N}\sum_{i=1}^{N}\mathrm{E}_0\left[\frac{\nu^2_{i0}(\epsilon_{i\,t}-1)^2}{\phi^2_0(z_\tau)}\right]+\frac{1}{N}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\mathrm{E}_0\left[\frac{\nu_{i0}\nu_{j0}(\epsilon_{i\,t}-1)(\epsilon_{j\,t}-1)}{\phi^2_0(z_\tau)}\right].
\end{aligned}\tag{13}
$$

By recalling that $\mathrm{E}_0[\epsilon_{i\,t}]=1$, we see that the first term of (13) is proportional to the variances of the innovations, while the second is proportional to their covariance. Under assumption C.2, the first term is bounded for any N while the second one diverges with N as there are $N(N-1)/2$ covariance terms. By assumption D, the second term of (13) is bounded for any N. Therefore, in large panels we require the cross–sectional dependence among the innovations to be weak. Similar conditions for the idiosyncratic components are made in the context of approximate factor models (see Bai and Ng (2002)). Assumption D is also supported by the empirical evidence (see Section

[2]We denote by $\partial/\partial\phi$ the Fréchet functional derivative.

7). Finally, analogous results are established by Li and Racine (2006) in additive panel models with a nonparametric component.

In order to carry out inference, the expressions in $V_{\boldsymbol{\xi}_o}(z_\tau)$ have to be replaced by their sample analogues. The integral of the squared kernel is a constant specific to the chosen kernel (e.g. it is 1 if we use a Gaussian kernel, $5/7$ for a quartic kernel). Then, using the estimators of the parameters $\widehat{a}_{iT}, \widehat{\boldsymbol{\delta}}_{iT}, \widehat{\nu}_{iT}$, given in Theorem 4 below, we define the estimated residuals

$$\widehat{\epsilon}_{i\,t} = \frac{x_{i\,t}}{\widehat{a}_{iT}\mu_{i\,t}(\mathcal{F}_{t-1}, \widehat{\boldsymbol{\delta}}_{iT})\widehat{\phi}_{NT\,\widehat{\boldsymbol{\xi}}_T}(z_t)}.$$

The localized Fisher information $\bar{i}_{N\,\boldsymbol{\xi}_o}(z_\tau)$ can be estimated, for any $z_\tau \in [0,1]$, by using the sample counterpart of (13):

$$\widehat{\bar{i}}_{NT\,\widehat{\boldsymbol{\xi}}_T}(z_\tau) \quad = \quad \frac{\sum_{t=1}^{T} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\left[\sum_{i=1}^{N} \frac{\widehat{\nu}_{iT}}{\widehat{\phi}_{NT\,\widehat{\boldsymbol{\xi}}_T}(z_t)}\left(\widehat{\epsilon}_{i\,t} - 1\right)\right]^2}{N\sum_{t=1}^{T} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)}. \tag{14}$$

Analogously, $\bar{j}_{N\,\boldsymbol{\xi}_o}(z_\tau)$, is estimated by

$$\widehat{\bar{j}}_{NT\,\widehat{\boldsymbol{\xi}}_T}(z_\tau) \quad = \quad \frac{\sum_{t=1}^{T} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\sum_{i=1}^{N} \frac{\widehat{\nu}_{iT}}{\widehat{\phi}^2_{NT\,\widehat{\boldsymbol{\xi}}_T}(z_t)}\left(2\,\widehat{\epsilon}_{i\,t} - 1\right)}{N\sum_{t=1}^{T} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)}. \tag{15}$$

Alternatively the denominators in (14) and (15) can be substituted by NTh_{NT}. Proofs of the consistency of these estimators are in Gasser and Müller (1979) and Müller (1984).

Prior to turning to the estimation of the parameters in the next section, we study the relation between them and the curve ϕ. Let us first consider the case $N = 1$, and focus on $\boldsymbol{\xi}_{1\,0}$. In the seminonparametric setting, the Fisher information matrix for $\boldsymbol{\xi}_{1\,0}$ is the sum of a parametric component minus a correction due to the presence of the curve:

$$\mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}^m_{1\,t}}{\partial \boldsymbol{\xi}_1}(\boldsymbol{\xi}_{1\,0},\phi_0)\frac{\partial \boldsymbol{\ell}^m_{1\,t}}{\partial \boldsymbol{\xi}_1^{\mathrm{T}}}(\boldsymbol{\xi}_{1\,0},\phi_0)\right] - \mathbf{v}\,\mathrm{E}_0\left[\left(\frac{\partial \boldsymbol{\ell}^m_{1\,t}}{\partial \phi}(\boldsymbol{\xi}_{1\,0},\phi_0)\right)^2\right]\mathbf{v}^{\mathrm{T}}, \tag{16}$$

where $\mathbf{v}$ is a generic vector of the same size as $\boldsymbol{\xi}_{1\,0}$. Note that, since the second term in (16) is positive definite, we have a smaller Fisher information with respect to the fully parametric case. We define a *least favorable direction* $\mathbf{v}^*$ as the minimizer of the seminonparametric Fisher information matrix (16) over all possible directions $\mathbf{v}$. Severini and Wong (1992) prove that an estimator of the curve having as tangent vector the least favorable direction, and called *least favorable curve*, delivers an unbiased estimator of the parameter $\boldsymbol{\xi}_{1\,0}$. It can be shown that the explicit form of the least favorable direction is

$$\mathbf{v}^* = -\frac{\mathrm{E}_0\left[\frac{\partial^2 \boldsymbol{\ell}^m_{1\,t}}{\partial \boldsymbol{\xi}_1 \partial \phi}(\boldsymbol{\xi}_{1\,0},\phi_0)\right]}{\mathrm{E}_0\left[\frac{\partial^2 \boldsymbol{\ell}^m_{1\,t}}{\partial \phi^2}(\boldsymbol{\xi}_{1\,0},\phi_0)\right]}. \tag{17}$$

Since all marginals depend on the curve, when $N > 1$, in (17) we have to substitute $\boldsymbol{\ell}^m_{1\,t}$ with $\sum_i \boldsymbol{\ell}^m_{i\,t}/N$. We denote the least favorable curve in a generic value of the parameters as $\phi_{\boldsymbol{\xi}}$ and we require it to satisfy the regularity assumption L in Appendix A. The tangent vector to this curve computed in the true value of the paramters, which is the first derivative of the least favorable curve with respect to the parameters, is then defined as

$$\phi'_{\boldsymbol{\xi}_o} = -\frac{\frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial^2 \boldsymbol{\ell}_{it}^m}{\partial \boldsymbol{\xi}\partial\phi}(\boldsymbol{\xi}_{i\,0},\phi_0)\right]}{\frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial^2 \boldsymbol{\ell}_{it}^m}{\partial\phi^2}(\boldsymbol{\xi}_{i\,0},\phi_0)\right]} \equiv -\frac{\bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_o}}{\bar{j}_{N\,\boldsymbol{\xi}_o}}. \tag{18}$$

In the following Theorem we show the asymptotic properties of the vector tangent to the estimated curve $\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}$.

Theorem 3 – Least Favorable Direction *Given the estimator $\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau)$ and under assumptions A, B, C.1, I, K, L, S in Appendix A, and if $NTh_{NT}\to\infty$ and $h_{NT}\to 0$ as $NT\to\infty$ we have:*

a) if $T\to\infty$ and N is finite, then, for any $z_\tau\in[0,1]$,

$$\widehat{\phi}'_{\boldsymbol{\xi}_o\,NT}(z_\tau)\stackrel{P}{\to}\phi'_{\boldsymbol{\xi}_o},$$

where $\phi'_{\boldsymbol{\xi}_o}$ is defined in (18)*;*

b) if $T\to\infty$ and $N\to\infty$ and assumption C.2 in Appendix A holds, then, for any $z_\tau\in[0,1]$,

$$\widehat{\phi}'_{\boldsymbol{\xi}_o\,NT}(z_\tau)\stackrel{P}{\to}\mathbf{0}.$$

When N is large the tangent vector to the curve becomes identically zero. Intuitively, this is the result of the fact that when $N\to\infty$ the nonparametric and the parametric components are asymptotically uncorrelated.

Last, we show the asymptotic properties of the estimated curve for any value for the parameters. Indeed, $\widehat{\boldsymbol{\xi}}_T$ and $\widehat{\phi}_{\boldsymbol{\xi}\,NT}$ depend on each other. To break this feedback loop, in the following Corollary we show that, for any given value of the parameters $\boldsymbol{\xi}$, the estimator defined in (12) is an estimator of a least favorable curve, a result which is necessary to prove consistency of the estimated parameters in Theorem 4. We prove the Corollary only for finite N as, when $N\to\infty$, the effect of the nuisance parameter ϕ becomes negligible (see part b of Theorem 3).

Corollary 1 – Least Favorable Curve *Under the same Assumptions of Theorem 3.a, for any $\boldsymbol{\xi}\in\Xi$ and any $z_\tau\in[0,1]$,*

$$\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)\stackrel{P}{\to}\phi_{\boldsymbol{\xi}}(z_\tau).$$

The first part of the proof of the Corollary relies on assumption N in Appendix A about the existence of a limiting curve (and its derivatives) for any value of the parameters. In a second part, we show that for any $\boldsymbol{\xi}$, the nonparametric estimator $\widehat{\phi}_{\boldsymbol{\xi}\,NT}$ converges to the least favorable curve $\phi_{\boldsymbol{\xi}}$, which is done in Lemma 4.

5.2 Estimation of the parameters

We now turn to the estimation of the marginals and copula parameters, $(\boldsymbol{\xi}_{1\,0},\ldots,\boldsymbol{\xi}_{N\,0})$ and $\boldsymbol{\psi}$. Since $\Xi_i\cap\Xi_j=\emptyset$ for $i\neq j$, estimation of $(\boldsymbol{\xi}_{1\,0},\ldots,\boldsymbol{\xi}_{N\,0})$ boils down to N independent optimizations of the marginal log–likelihoods,

$$\widehat{\boldsymbol{\xi}}_{i\,T} = \arg\max_{\boldsymbol{\xi}_i\in\Xi_i}\sum_{t=1}^{T}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_i,\widehat{\phi}_{\boldsymbol{\xi}\,NT}),\quad i=1,\ldots,N. \tag{19}$$

The asymptotic properties are proved in the following Theorem.

Theorem 4 – Parameters of the Marginals *Consider the estimator of a least favorable curve in* (12) *and for any* $i = 1, \dots, N$, *let* $\widehat{\boldsymbol{\xi}}_{iT}$ *be the vector of parametric estimates in* (19), *then under assumptions A, B, C.1, I, L, P, S, in Appendix A, we have*

a) if $T \to \infty$, *then* $\widehat{\boldsymbol{\xi}}_{iT} \xrightarrow{P} \boldsymbol{\xi}_{i0}$;

b) if $T \to \infty$ *and* N *is finite, then*

$$\sqrt{T}\left(\widehat{\boldsymbol{\xi}}_T - \boldsymbol{\xi}_0\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, (\mathbf{H}^*_{\boldsymbol{\xi}_o})^{-1}\, \mathbf{I}^*_{\boldsymbol{\xi}_o}\, (\mathbf{H}^*_{\boldsymbol{\xi}_o})^{-1}\,\right),$$

where

$$\begin{aligned}
\mathbf{I}^*_{\boldsymbol{\xi}_o} &= \begin{pmatrix} \mathcal{I}_{\boldsymbol{\xi}_{1o}\boldsymbol{\xi}_{1o}} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{I}_{\boldsymbol{\xi}_{No}\boldsymbol{\xi}_{No}} \end{pmatrix} - \left(\bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_o}\bar{\boldsymbol{d}}^T_{N\boldsymbol{\xi}_o}\right) \otimes \frac{\bar{i}_{N\boldsymbol{\xi}_o}}{\bar{j}^2_{N\boldsymbol{\xi}_o}}, \\
\mathbf{H}^*_{\boldsymbol{\xi}_o} &= \begin{pmatrix} \mathcal{H}_{\boldsymbol{\xi}_{1o}\boldsymbol{\xi}_{1o}} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{H}_{\boldsymbol{\xi}_{No}\boldsymbol{\xi}_{No}} \end{pmatrix} - \left(\bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_o}\bar{\boldsymbol{d}}^T_{N\boldsymbol{\xi}_o}\right) \otimes \frac{1}{\bar{j}_{N\boldsymbol{\xi}_o}},
\end{aligned}$$

where $\bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_o}$ *is the numerator of* (18), $\bar{j}_{N\boldsymbol{\xi}_o}$ *is defined in Theorem 2, and, for any* $i = 1, \dots, N$,

$$\mathcal{I}_{\boldsymbol{\xi}_{io}\boldsymbol{\xi}_{io}} = \mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}^m_{it}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i0}, \phi_0)\frac{\partial \boldsymbol{\ell}^m_{it}}{\partial \boldsymbol{\xi}^T_i}(\boldsymbol{\xi}_{i0}, \phi_0)\right], \quad \mathcal{H}_{\boldsymbol{\xi}_{io}\boldsymbol{\xi}_{io}} = -\mathrm{E}_0\left[\frac{\partial^2 \boldsymbol{\ell}^m_{it}}{\partial \boldsymbol{\xi}_i \partial \boldsymbol{\xi}^T_i}(\boldsymbol{\xi}_{i0}, \phi_0)\right].$$

c) if $T \to \infty$ *and* $N \to \infty$ *and under assumptions C.2 and D in Appendix A, then*

$$\sqrt{T}\left(\widehat{\boldsymbol{\xi}}_T - \boldsymbol{\xi}_0\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \mathbf{H}^{-1}_{\boldsymbol{\xi}_o}\, \mathbf{I}_{\boldsymbol{\xi}_o}\, \mathbf{H}^{-1}_{\boldsymbol{\xi}_o}\,\right),$$

where

$$\mathbf{I}_{\boldsymbol{\xi}_o} = \begin{pmatrix} \mathcal{I}_{\boldsymbol{\xi}_{1o}\boldsymbol{\xi}_{1o}} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{I}_{\boldsymbol{\xi}_{No}\boldsymbol{\xi}_{No}} \end{pmatrix}, \quad \mathbf{H}_{\boldsymbol{\xi}_o} = \begin{pmatrix} \mathcal{H}_{\boldsymbol{\xi}_{1o}\boldsymbol{\xi}_{1o}} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{H}_{\boldsymbol{\xi}_{No}\boldsymbol{\xi}_{No}} \end{pmatrix}.$$

For N fixed, consistency and asymptotic Gaussianity of the estimated parameters hold, provided we have a consistent estimator of ϕ_0 satisfying assumption N. Theorem 3 guarantees that this is the case for $\widehat{\phi}_{\boldsymbol{\xi}\, NT}$ even when $N \to \infty$. Notice that the parameter estimation of the marginals is based on N univariate maximizations, thus N plays no role in computing the first term of $\mathbf{I}^*_{\boldsymbol{\xi}_o}$ and $\mathbf{H}^*_{\boldsymbol{\xi}_o}$. The asymptotic covariance matrix is larger than the parametric lower bound. This is due to two reasons. First, since we use the marginals to estimate $\boldsymbol{\xi}_0$, we obtain the usual sandwich form as in Quasi Maximum Likelihood estimation. Second, as explained in the previous section, the presence of the curve modifies both the information matrix and the Hessian, and it is straightforward to see that the asymptotic covariance matrix is greater than or equal to the seminonparametric lower bound, i.e. $(\mathbf{H}^*_{\boldsymbol{\xi}_o})^{-1}\mathbf{I}^*_{\boldsymbol{\xi}_o}(\mathbf{H}^*_{\boldsymbol{\xi}_o})^{-1} \succeq (\mathbf{I}^*_{\boldsymbol{\xi}_o})^{-1}$. Notice that the correction term $\bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_o}\bar{\boldsymbol{d}}^T_{N\boldsymbol{\xi}_o}$ is not block diagonal, as the curve is contained in all the marginal distributions. Also by Theorem 3 this correction term is $O(N^{-2})$. Thus, if we let $N \to \infty$ this term becomes negligible and the seminonparametric asymptotic covariance converges to its parametric counterpart, i.e. to $\mathbf{H}^{-1}_{\boldsymbol{\xi}_o}\, \mathbf{I}_{\boldsymbol{\xi}_o}\, \mathbf{H}^{-1}_{\boldsymbol{\xi}_o}$.

In short, we see the advantage of inference from the marginals. Estimation reduces to a simple iterative process between a closed form estimator, $\widehat{\phi}_{\boldsymbol{\xi}\,NT}$, and N univariate optimizations with respect to five parameters each ($\boldsymbol{\xi}_i = (a_i, \alpha_i, \beta_i, \gamma_i, \nu_i)$).

Finally, given the estimators $\widehat{\phi}_{\boldsymbol{\xi}\,NT}$ and $\widehat{\boldsymbol{\xi}}_T = (\widehat{\boldsymbol{\xi}}^{\mathrm{T}}_{1\,T}, \ldots, \widehat{\boldsymbol{\xi}}^{\mathrm{T}}_{N\,T})^{\mathrm{T}}$ obtained in (12) and (19), we estimate $\boldsymbol{\psi}_0$ by maximizing the copula log–likelihood:

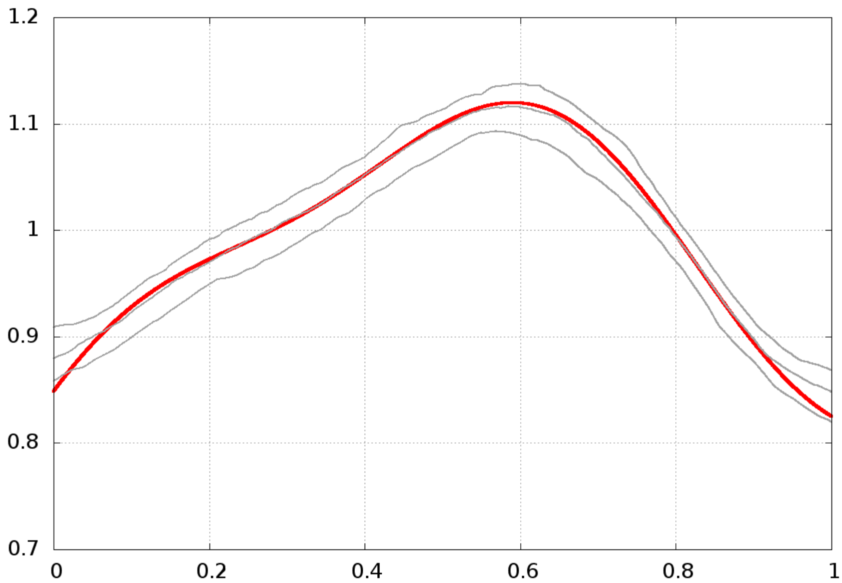
$$\widehat{\boldsymbol{\psi}}_T = \arg\max_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \sum_{t=1}^{T} \ell_t^c(\widehat{\boldsymbol{\xi}}_T, \boldsymbol{\psi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}). \tag{20}$$

Consistency of this estimator is proved in the following Theorem.

Theorem 5 – Parameters of the Copula *Consider the estimator of a least favorable curve in* (12) *and the marginals' parameters estimators* $\widehat{\boldsymbol{\xi}}_{i\,T}$*, for* $i = 1, \ldots, N$*, in* (19)*, let* $\widehat{\boldsymbol{\psi}}_T$ *be the vector of estimators in* (20)*, then under assumptions A, B, C.1, I, L, P in Appendix A, if* $T \to \infty$*,* $\widehat{\boldsymbol{\psi}}_T \xrightarrow{P} \boldsymbol{\psi}_0$.

6 Monte Carlo Study

Figure 2: Simulation Study: Common Trend.



The figure shows the profile of the common trend (thick line) used in the Monte Carlo simulations together with the 95% quantile range and the median of the simulated distribution of the trend estimator (thin lines).

We carry out a simulation study to investigate the finite sample properties of the SPVMEM components estimators and to assess the reliability of the asymptotic standard errors estimators for inference. The simulation exercise can be carried out according a structural or a reduced form approach, and in this section we opt for the latter option. The structural approach consists of simulating a panel of high frequency prices from a continuous time model with stochastic

Table 3: Simulation Study: Nonparametric Component Estimator.

z_τ	simulated squared bias (× 100)	simulated variance (× 100)	simulated average est. variance (× 100)	simulated 90% CI coverage
0.17	0.0005	0.0123	0.0087	0.8322
0.33	0.0015	0.0131	0.0112	0.8260
0.50	0.0008	0.0159	0.0128	0.8406
0.67	0.0021	0.0138	0.0121	0.8340
0.83	0.0007	0.0117	0.0087	0.7521

The table reports the squared bias, variance, average estimated variance and coverage of the 90% confidence interval of the nonparametric common trend estimator in different points of the support of the trend.

Table 4: Simulation Study: Parametric Component Estimator.

	simulated squared bias (× 100)	simulated variance (× 100)	simulated average est. variance (× 100)	simulated 90% CI coverage
a_i	0.0028	0.0914	0.0927	0.9079
α_i	0.0000	0.0055	0.0069	0.9384
γ_i	0.0001	0.0143	0.0171	0.9056
β_i	0.0010	0.0461	0.0476	0.8958
ν_i	0.0002	0.0293	0.0216	0.8945

The table reports the squared bias, variance, average estimated variance and coverage of the 90% confidence interval of the marginal parameter estimator.

volatility and leverage, constructing the high frequency returns r_{itj} by discretely sampling such process and then finally constructing the realized measures x_{it}. On the other hand, the reduced form approach we follow consists of simulating x_{it} directly from the SPVMEM model. Direct simulation of the SPVMEM requires realizations of the sign of the past daily returns of each asset. Rather than simulating the daily return process, we assume the sign of daily return to be iid Bernoulli random deviates with equi probability of being positive or negative and we simulate them as such. The coefficients of the idiosyncratic components of the SPVMEM are chosen to reproduce approximately the empirical characteristics of the data: α_i, γ_i and β_i are set respectively to 0.05, 0.06 and 0.90 for each series. The scale factors a_i and the marginal variances $1/\nu_i$ of are drawn from an Exponential distribution. The plot of the common trend component $\phi(z_t)$ used in the simulations is displayed in Figure 2. We also allow for weak cross-sectional dependence in the innovations $\epsilon_{i\,t}$. We induce dependence as follows. For each replication of the Monte Carlo, we simulate a covariance matrix from a Wishart distribution of order N. The parameters of the Wishart are chosen in way such that its expectation is the identity and the standard deviation of the off diagonal elements is 0.03. We then construct the correlation matrix by appropriately standardizing the covariance matrix and simulate a panel of T deviates from a multivariate normal with mean zero and covariance matrix equal to the simulated correlation matrix. The normal deviates are then mapped into uniforms random variables through the normal cdf. Finally, we plug each of the N uniform deviates series into the inverse cdf of Gamma distribution with dispersion equal to $1/\nu_i$. The simulation procedure ensures that marginally the innovations $\epsilon_{i\,t}$ have a Gamma distribution and exhibit cross–sectional dependence. The Monte Carlo experiment is replicated for 1‘000 times. The dimensions of the panel are equal to $N = 100$ and $T = 5000$, which mimics the dimensions of the S&P100 panel analysed in this paper.

Table 5: SPDR Estimated Parameters

	SPVMEM						MEM					
	a_i	α_i	γ_i	β_i	ν_i	π_i	a_i	α_i	γ_i	β_i	ν_i	π_i
XLB	0.32 (0.033)	0.22 (0.018)	0.13 (0.012)	0.57 (0.026)	0.33 (0.000)	0.86 (0.032)	0.05 (0.012)	0.25 (0.037)	0.13 (0.027)	0.66 (0.035)	0.33 (0.031)	0.97 (0.053)
XLE	0.13 (0.018)	0.26 (0.015)	0.09 (0.013)	0.66 (0.016)	0.34 (0.000)	0.96 (0.023)	0.10 (0.023)	0.25 (0.032)	0.12 (0.027)	0.64 (0.036)	0.34 (0.030)	0.95 (0.050)
XLF	0.22 (0.017)	0.29 (0.011)	0.12 (0.012)	0.54 (0.015)	0.60 (0.000)	0.90 (0.019)	0.02 (0.012)	0.27 (0.057)	0.13 (0.062)	0.67 (0.045)	0.61 (0.066)	1.00 (0.079)
XLI	0.12 (0.013)	0.27 (0.012)	0.08 (0.009)	0.64 (0.015)	0.47 (0.000)	0.94 (0.019)	0.02 (0.009)	0.29 (0.033)	0.11 (0.023)	0.66 (0.032)	0.48 (0.032)	1.00 (0.047)
XLK	0.09 (0.010)	0.24 (0.010)	0.10 (0.006)	0.69 (0.010)	0.69 (0.000)	0.98 (0.014)	0.04 (0.018)	0.27 (0.045)	0.13 (0.027)	0.66 (0.045)	0.70 (0.044)	0.99 (0.065)
XLP	0.04 (0.004)	0.16 (0.005)	0.07 (0.006)	0.78 (0.005)	0.77 (0.000)	0.98 (0.008)	0.02 (0.011)	0.19 (0.045)	0.11 (0.044)	0.74 (0.039)	0.75 (0.065)	0.99 (0.063)
XLU	0.10 (0.006)	0.25 (0.011)	0.05 (0.009)	0.69 (0.010)	0.59 (0.000)	0.97 (0.016)	0.02 (0.012)	0.27 (0.052)	0.06 (0.042)	0.70 (0.040)	0.65 (0.041)	1.00 (0.069)
XLV	0.11 (0.008)	0.37 (0.013)	0.10 (0.009)	0.52 (0.012)	1.15 (0.000)	0.94 (0.018)	0.02 (0.014)	0.39 (0.064)	0.12 (0.043)	0.55 (0.052)	1.22 (0.048)	1.00 (0.085)
XLY	0.24 (0.019)	0.19 (0.011)	0.12 (0.011)	0.61 (0.019)	0.46 (0.000)	0.86 (0.022)	0.02 (0.008)	0.25 (0.036)	0.13 (0.034)	0.68 (0.035)	0.48 (0.038)	0.99 (0.053)

Estimated parameters and standard errors (in parenthesis) for the SPvMEM (left) and the univariate MEM (right).

Tables 3 and 4 summarize the results of the full set of Monte Carlo replications. Table 3 shows the Monte Carlo squared bias and variance of the trend estimator evaluated in five different points of the support of the curve. The table also reports the average of the squared estimated asymptotic standard error and the coverage rate of the asymptotic 90% confidence interval. The 90% asymptotic confidence interval is constructed as the trend point estimate plus or minus the normal quantile multiplying the estimated asymptotic standard error. We also report in Figure 2 the 95% quantile range and the median of the simulated distribution of the nonparametric trend estimator. Results show that the nonparametric estimator behaves satisfactorily and that the estimator is rather precise. The average estimated standard error is close to its target, although the simulations suggest that the estimator is slightly downward biased. The downward bias of the standard error also affects inference. In fact, the coverage rate of the 90% confidence interval has coverage probability close to the nominal level but is slightly smaller. Table 4 reports analog summary statistics for the parametric part of the model. Note that the table reports summary averages across all Monte Carlo replications and all series in the panel. Again, the estimation procedure delivers satisfactory estimates. The average estimate standard errors closely tracks their population analogs and the coverage rate of the 90% confidence interval closely matches the desired level.

Overall, results show that the proposed estimation procedure performs well in moderately large panels and that the large sample standard error estimators provide adequate inference.

7 Empirical Analysis

In this section we apply the SPvMEM to analyse the SPDR and S&P100 panels. Section 7.1 presents the estimation results over the full sample, while Section 7.2 presents the results of a forecasting exercise where the model is compared against a number of alternative specifications.

7.1 In–Sample Estimation Results

The estimation of the SPvMEM model (3) on the SPDR and S&P100 panels is carried out using a quartic kernel with a bandwidth resulting in a trend computed over a three month window. Esti-

Table 6: S&P100 Estimated Parameters

		SPVMEM						MEM					
		a_i	α_i	γ_i	β_i	ν_i	π_i	a_i	α_i	γ_i	β_i	ν_i	π_i
Disc	$q_{0.25}$	0.11	0.27	0.07	0.60	0.26	0.95	0.06	0.27	0.07	0.62	0.27	0.98
	$q_{0.50}$	0.16	0.30	0.09	0.62	0.31	0.97	0.07	0.29	0.09	0.66	0.32	0.98
	$q_{0.75}$	0.24	0.31	0.10	0.66	0.37	0.98	0.09	0.31	0.10	0.67	0.37	0.99
Ener	$q_{0.25}$	0.10	0.25	0.06	0.65	0.23	0.97	0.09	0.25	0.07	0.65	0.24	0.97
	$q_{0.50}$	0.12	0.26	0.08	0.68	0.32	0.98	0.12	0.26	0.08	0.68	0.34	0.98
	$q_{0.75}$	0.21	0.28	0.09	0.70	0.53	0.98	0.14	0.28	0.09	0.70	0.54	0.99
Fin	$q_{0.25}$	0.10	0.30	0.09	0.53	0.25	0.97	0.05	0.30	0.08	0.56	0.26	0.99
	$q_{0.50}$	0.12	0.35	0.11	0.57	0.30	0.98	0.05	0.35	0.11	0.58	0.31	0.99
	$q_{0.75}$	0.16	0.37	0.13	0.64	0.43	0.98	0.07	0.38	0.13	0.64	0.45	1.00
Heal	$q_{0.25}$	0.09	0.27	0.06	0.58	0.35	0.97	0.07	0.27	0.06	0.56	0.35	0.97
	$q_{0.50}$	0.14	0.31	0.07	0.63	0.46	0.98	0.08	0.31	0.09	0.63	0.46	0.98
	$q_{0.75}$	0.17	0.37	0.09	0.66	0.69	0.98	0.13	0.37	0.10	0.66	0.66	0.99
Ind	$q_{0.25}$	0.10	0.28	0.08	0.58	0.25	0.96	0.06	0.28	0.08	0.58	0.26	0.98
	$q_{0.50}$	0.12	0.33	0.09	0.60	0.28	0.97	0.08	0.33	0.09	0.61	0.29	0.98
	$q_{0.75}$	0.16	0.34	0.12	0.63	0.36	0.98	0.10	0.34	0.11	0.63	0.37	0.99
Mat	$q_{0.25}$	0.17	0.26	0.09	0.59	0.24	0.95	0.09	0.27	0.09	0.59	0.24	0.97
	$q_{0.50}$	0.18	0.29	0.10	0.61	0.26	0.96	0.13	0.29	0.10	0.63	0.27	0.97
	$q_{0.75}$	0.27	0.32	0.10	0.65	0.32	0.96	0.17	0.32	0.10	0.66	0.33	0.98
Stap	$q_{0.25}$	0.07	0.24	0.05	0.61	0.31	0.97	0.05	0.24	0.05	0.59	0.31	0.97
	$q_{0.50}$	0.10	0.30	0.06	0.62	0.39	0.97	0.07	0.31	0.07	0.63	0.39	0.98
	$q_{0.75}$	0.14	0.33	0.07	0.70	0.56	0.98	0.11	0.34	0.08	0.71	0.56	0.98
Tech	$q_{0.25}$	0.11	0.26	0.08	0.49	0.22	0.96	0.06	0.27	0.08	0.51	0.24	0.97
	$q_{0.50}$	0.21	0.29	0.10	0.61	0.27	0.97	0.09	0.31	0.10	0.60	0.28	0.99
	$q_{0.75}$	0.28	0.39	0.12	0.68	0.37	0.99	0.14	0.40	0.12	0.68	0.38	0.99
Util	$q_{0.25}$	0.09	0.27	0.05	0.62	0.30	0.97	0.07	0.28	0.05	0.62	0.31	0.98
	$q_{0.50}$	0.11	0.29	0.09	0.65	0.36	0.98	0.08	0.30	0.09	0.65	0.36	0.98
	$q_{0.75}$	0.13	0.31	0.10	0.68	0.56	0.98	0.10	0.31	0.10	0.67	0.56	0.98

Estimated parameters for the SPvMEM (left) and the univariate MEM (right). The table reports the median, 1^{st} quartile and 3^{rd} quartile of the parameter estimates across the each industry group.

mation details are provided in the QQQ Web version of the paper with SPvMEM results reported on the left panels of Table 5 (individual series of SPDR); for the S&P100 panel the results are conveniently aggregated by quantiles across industry groups in Table 12 and by individual ticker in Table QQQ. Asymmetric MEM(1,1)

$$x_{it} = a_i \mu_{it} \epsilon_{it} = \tilde{\mu}_{it} \epsilon_{it} = \left(\omega_i + (\alpha_i + \gamma_i \mathbf{1}_{\{r_{i\,t-1}<0\}}) x_{it-1} + \beta_i \tilde{\mu}_{it-1} \right) \epsilon_{it}$$

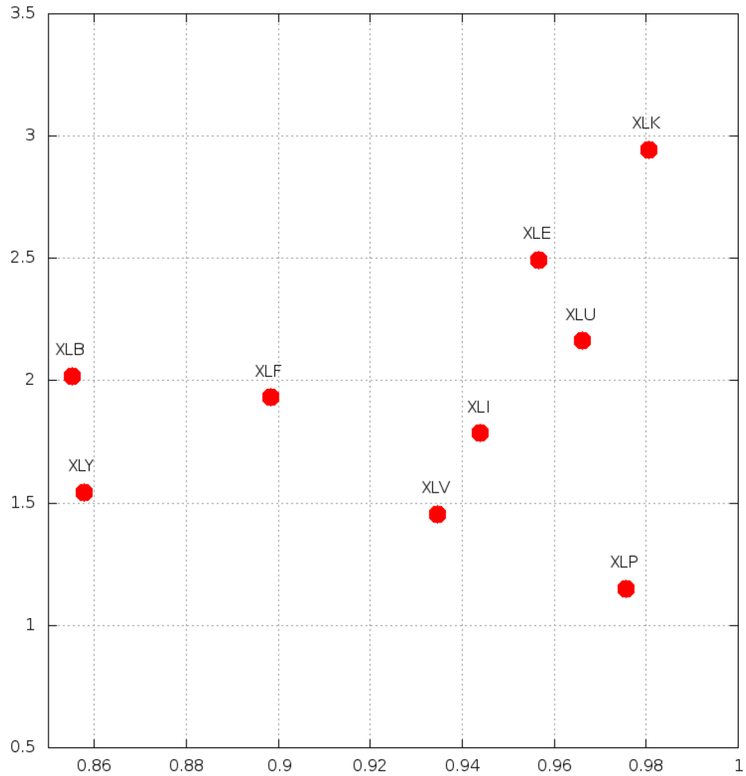
provide estimation benchmark results (on the right panels of the same tables in the Web version).

When contrasted to typical GARCH estimates, the values of the α's and γ's are higher while they are lower for the β's, as a result of a better informative content of realized measures as estimates of volatility (see Brownlees and Gallo (2010) and Shephard and Sheppard (2010) for similar evidence). As customary, we have positive asymmetric reaction to negative news. The estimated persistence $\alpha_i + \gamma_i/2 + \beta_i$ reveals important differences across assets. For SPDR the sectors with higher persistence are Consumer Staples and Technology, meaning that these are the sectors with longer lasting idiosyncratic departures from the systematic component. For S&P100 the differences between the least and the most persistent idiosyncratic components are wide and, in general, they appear to be higher than the ones in SPDR. We interpret this as the result that individual assets have a higher level of idiosyncrasy in comparison to sectoral ETFs. The systematic scale factors are fairly close to the sample average volatilities (see Table 1). Last, the estimates of the Gamma distribution also differ substantially across sectors and assets, implying differences in the marginal distributions. By contrast, in the vast majority of cases, the persistence implied by the MEMs are essentially equal to one, hinting at the presence of integrated dynamics (violation of the stationarity condition), and the unconditional volatilities (denoted by a_i for comparison purposes with our model) are often far (especially for SPDR) from the sample average volatilities.

Figure 3 displays the scatter plot of the estimated (log) unconditional mean versus persistence for the SPvMEM in the SPDR panel. Technology is the sector with the highest persistence and unconditional volatility, followed by Energy and Utilities. The Industrial, and Health Care and Consumer Staple sectors have lower levels of volatility but still have persistent dynamics while Consumer Discretionary, Materials, and Financials, have lower levels of volatility and low persistence. As far as the univariate MEM results are concerned (not reported in the plot), the estimated $\alpha_i + \gamma_i/2 + \beta_i$ collapse to unity as a consequence of the strong volatility persistence, meaning that it is essentially the same in all sectors regardless of the unconditional level of volatility. Moreover, the model implied estimate of unconditional volatility is often excessively large as a consequence of the fact that persistence is close to unity.

Figure 4 shows (from top to bottom) the estimated mean $\sqrt{252\, a_i \phi(z_t) \mu_{i\,t}}$, the systematic component $\sqrt{\phi(z_t)}$, and the idiosyncratic components $\sqrt{252\, a_i \mu_{i\,t}}$ for SPDR (left) and S&P100 (right). The fitted mean series accurately track the movements of realized volatilities. The systematic volatility for both panels are essentially close with minor differences for the 2002 burst in volatility recorded differently in the two sets of data (averages vs individual stocks). The plot of the systematic volatility components also displays the (pointwise) 95% confidence bands. Note that, especially in the S&P100 panel, the width of interval is rather tight and the bands are hard to see. The horizontal line at one is used as a benchmark to identify periods of systematic risk amplification and contraction. The systematic trend can be interpreted in terms of the underlying movements in the business cycle and financial markets. We trace the increase in volatility following the market drop in mid 2002 with a cut to a half to its unconditional value during the years of volatility moderation and the sudden increase to three times its normal value toward the last quarter of 2008. The idiosyncratic volatilities are stationary and vary around the unconditional means.

Figure 3: SPDR Persistence versus volatility



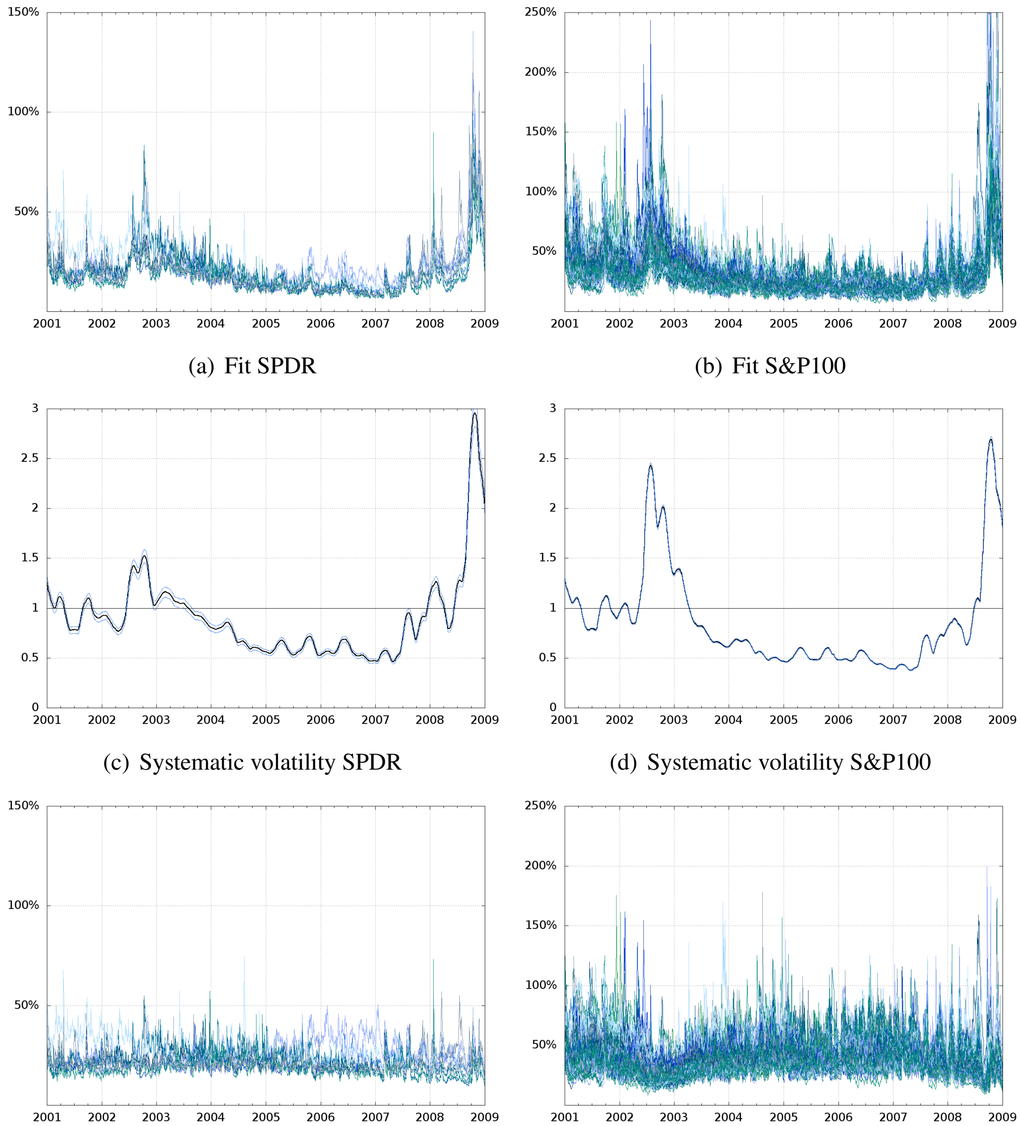
Scatter plots of persistence (X-axis) versus unconditional volatility (Y-axis) for the SPvMEM.

To get deeper insights, Figure 5 displays the idiosyncratic volatilities of the Energy, Financial and Technology sectors only, which allow to visually identify periods of distress around the systematic trend. At the beginning of 2001, Technology was the most volatile sector due to the aftermath of the burst of the dot com bubble. Between 2005 and 2007, concerns about oil prices generated an increased level of uncertainty in the Energy sector. Finally, the Financial sector had a surge in volatility starting from July 2007 with the beginning of the credit crunch.

Table 7 reports the sample correlation matrix $\mathbf{R}$ as defined in Section 3 for the SPDR while Figure 6 provides analog estimate for the S&P100 in a heat map. The average correlations are around 0.40 and 0.25 in the SDPR and S&P100 panels respectively. In comparison to the original data, we see that the common trend captures a significant amount of cross–sectional dependence. The heat map of the S&P100 unveils clustering among stocks that belong to the same sector that are not visible from the raw data. This is the case for Technology, Financials, Energy and Utilities.

Last, we find that residuals of the model do not exhibit significant serial dependence. Table 8 reports the estimates of the lag one autocorrelation matrix for the SPDR while Figure 7 provides the lag one autocorrelation matrix for S&P100 in a heat map. The autocorrelations are negligible and there is virtually no significant evidence. Also, the cross autocorrelations are exiguous and we find no systematic spillover pattern across series. Overall, the cross autocorrelations are small and no dynamic pattern is detected.

Figure 4: Volatility decomposition



(a) Fit SPDR

(b) Fit S&P100

(c) Systematic volatility SPDR

(d) Systematic volatility S&P100

(e) Idiosyncratic volatilities SPDR

(f) Idiosyncratic volatilities S&P100

The top row shows the estimated fit of the annualized volatilities entailed by the model: $\sqrt{252\ a_i\ \phi(z_t)\ \mu_{i\,t}}$. The middle row shows the systematic volatility $\sqrt{\phi(z_t)}$, and the bottom row shows the annualized idiosyncratic volatilities $\sqrt{252\ a_i\ \mu_{i\,t}}$. Left column is for SPDR and right column for S&P100.

7.2 Out–of–Sample Forecasting Analysis

To assess the predictive ability of our proposed specification we carry out a forecasting exercise. We compare the SPvMEM against four alternative specifications: a MEM(1,1), a MEM(2,2), a Composite Likelihood MEM(1,1) (CL–MEM(1,1), cf. Engle *et al.* (2008)) and a (univariate) Seminonparametric MEM (SPMEM). All the MEM specifications considered allow for asymmetric reactions to past negative returns. In detail:

Table 7: SPDR Residual Copula Dependence

	XLB	XLE	XLF	XLI	XLK	XLP	XLU	XLV	XLY
XLB	1								
XLE	0.37	1							
XLF	0.29	0.25	1						
XLI	0.44	0.36	0.44	1					
XLK	0.27	0.27	0.36	0.30	1				
XLP	0.25	0.21	0.21	0.23	0.28	1			
XLU	0.33	0.30	0.20	0.28	0.22	0.19	1		
XLV	0.26	0.22	0.26	0.26	0.51	0.42	0.22	1	
XLY	0.37	0.28	0.45	0.41	0.36	0.26	0.25	0.29	1

The table reports the estimated Gaussian meta-copula correlation matrix for the SPDR panel.

Table 8: SPDR Residual Lag 1 Autocorrelation

	XLB	XLE	XLF	XLI	XLK	XLP	XLU	XLV	XLY
XLB	-0.00	0.03	0.03	0.03	0.04	0.02	0.09	0.02	0.00
XLE	0.03	-0.02	0.02	0.03	0.02	0.07	0.04	0.02	0.00
XLF	-0.01	0.03	-0.01	-0.00	0.05	0.03	0.06	0.01	0.00
XLI	0.04	0.03	0.01	-0.02	0.03	0.03	0.06	0.03	-0.00
XLK	0.04	-0.02	0.01	0.02	-0.02	0.00	0.04	-0.01	0.00
XLP	0.03	0.03	0.01	0.04	0.03	-0.01	0.06	-0.01	0.02
XLU	0.03	0.01	0.02	0.04	0.02	0.05	0.01	0.03	0.03
XLV	0.01	0.00	0.01	0.02	0.01	0.04	0.04	-0.03	-0.00
XLY	0.04	0.04	0.02	0.00	0.05	0.03	0.09	0.03	-0.02

The table reports the residual lag one autocorrelation matrix for the SPDR panel.

Table 9: SPDR Out–of–sample Forecasting

	SPVMEM	MEM(1,1)	MEM(2,2)	CL-MEM(1,1)	SPMEM
XLB	69.18	67.79	69.05	86.06	73.81
XLE	62.78	64.05	64.04	75.27	68.26
XLF	89.53	86.84	86.10	121.47	86.55
XLI	75.45	77.37	168.06	131.10	81.88
XLK	100.84	101.87	226.81	182.50	102.30
XLP	116.90	120.10	123.27	152.77	126.64
XLU	86.41	92.91	184.00	146.47	94.28
XLV	130.73	142.53	148.27	175.00	145.37
XLY	85.49	88.75	89.09	125.27	93.17

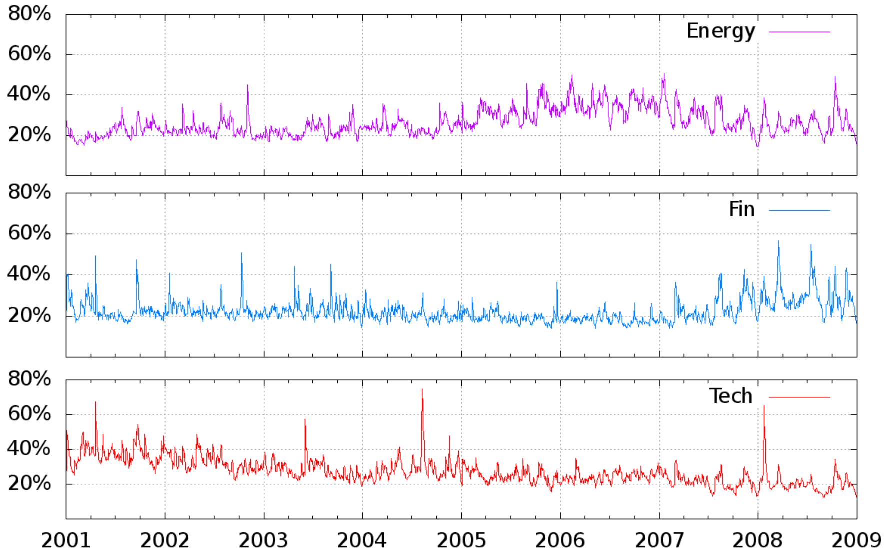
Out–of–sample QL losses for the SPvMEM, MEM(1,1), MEM(2,2), Composite Likelihood MEM(1,1) and (univariate) SPMEM.

Table 10: S&P100 Out–of–sample Forecasting

		SPVMEM	MEM(1,1)	MEM(2,2)	CL-MEM(1,1)	SPMEM
Disc	Mean	65.47	66.11	79.56	105.96	66.70
	q_{10}	56.12	56.79	56.78	78.37	55.95
	q_{90}	75.84	76.39	132.10	143.99	78.39
Ener	Mean	67.81	73.22	135.90	116.34	69.57
	q_{10}	47.61	47.24	45.51	61.28	47.72
	q_{90}	117.24	150.49	346.55	259.27	124.07
Fin	Mean	75.98	77.45	107.68	108.34	79.24
	q_{10}	59.25	58.97	58.36	83.35	61.40
	q_{90}	97.28	101.22	181.95	138.90	102.77
Heal	Mean	82.37	91.09	97.88	122.59	84.18
	q_{10}	57.07	68.79	75.65	97.19	58.11
	q_{90}	100.57	119.03	136.95	152.94	102.12
Ind	Mean	67.48	70.31	90.76	108.16	68.44
	q_{10}	57.63	57.01	55.55	82.67	58.80
	q_{90}	79.73	97.67	173.82	135.09	81.51
Mat	Mean	68.07	68.76	64.54	91.43	68.91
	q_{10}	57.55	58.54	54.90	75.37	58.43
	q_{90}	73.79	76.91	70.83	112.36	75.86
Stap	Mean	80.51	81.25	95.05	114.22	81.81
	q_{10}	62.16	62.48	58.28	74.34	60.61
	q_{90}	94.81	94.46	141.41	134.90	95.66
Tech	Mean	72.44	71.94	79.27	127.07	72.51
	q_{10}	58.33	57.68	54.12	95.50	55.80
	q_{90}	93.34	93.13	128.50	165.02	92.98
Util	Mean	70.63	70.47	91.54	107.64	72.27
	q_{10}	57.86	56.27	52.07	82.07	59.17
	q_{90}	89.70	91.28	129.84	144.28	88.45

Out–of–sample QL losses for the SPvMEM, MEM(1,1), MEM(2,2), Composite Likelihood MEM(1,1) and (univariate) SPMEM. The table reports the average loss, the 10% quantile and 90% quantile of each industry group.

Figure 5: Idiosyncratic volatilities for Energy, Financial and Technology



The figure shows the idiosyncratic volatility for Energy (top), Financials (middle) and Technology (bottom).

1. MEM without trend: $x_{it} = a_i \mu_{it} \epsilon_{it} = \tilde{\mu}_{it} \epsilon_{it}$, with $\tilde{\mu}_{it}$ specified as

   (a) MEM(1,1): $\omega_i + (\alpha_i + \gamma_i \mathbf{1}_{\{r_{i\,t-1}<0\}}) x_{it-1} + \beta_i \tilde{\mu}_{it-1}$

   (b) MEM(2,2): $\omega_i + (\alpha_{1i} + \gamma_i \mathbf{1}_{\{r_{i\,t-1}<0\}}) x_{it-1} + \beta_{1i} \tilde{\mu}_{it-1} + \alpha_{2i} x_{it-2} + \beta_{2i} \tilde{\mu}_{it-2}$

   (c) CL–MEM(1,1): $\omega + (\alpha + \gamma \mathbf{1}_{\{r_{i\,t-1}<0\}}) x_{it-1} + \beta \tilde{\mu}_{it-1}$

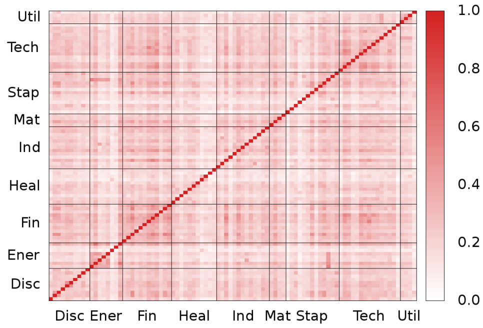
2. SPMEM(1,1) (individual trend): $x_{it} = a_i \phi_i(z_t) \mu_{it} \epsilon_{it} = \phi_i(z_t) \tilde{\mu}_{it} \epsilon_{it}$, with $\tilde{\mu}_{it}$ specified as in MEM(1,1) and $\phi_i(z_t)$ an asset specific, univariate version of expression 7.

Models without trend are the MEM(1,1), seen above, possibly extended in the specification to accommodate a second lag (MEM(2,2). Since parameter coefficients estimated over a panel of financial time series may cluster, the CL–MEM(1,1) consists of assuming parameters to be constant across series. The last specification allows volatility trends to be series specific and hence allows for assessing what the benefits are of estimating the common trend by pooling series.

The forecasting exercise is designed as follows. Starting from the beginning of 2007 we produce the series of one step ahead forecasts for each model using parameter estimates updated on the last weekday of each week. The prediction of the nonparametric trend is constructed by keeping constant the last estimate for the forecast horizon. The series of forecasts produced by the five approaches are evaluated using the QL loss function (Patton (2010)).

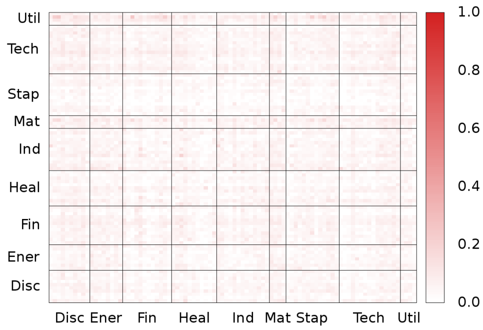
The choice of the kernel and the bandwidth for the out–of–sample exercise require further details. First, building up on Gijbels *et al.* (1999) among others, we fit the SPvMEM using a one sided quartic kernel for the forecasting application. Second, the bandwidth is chosen using an out–of–sample cross–validation criterion. We consider different bandwidths corresponding to different window lengths varying from one month to six months. We then estimate the models

Figure 6: S&P100 Residual Copula Dependence



The heatmap displays the estimated Gaussian meta-copula correlation matrix for the S&P100 panel.

Figure 7: S&P100 Residual Lag 1 Autocorrelation



The heatmap displays the residual lag one autocorrelation matrix for the S&P100 panel.

from the beginning of the sample until June 2006 using the different bandwidths and choose the bandwidth which delivers the one step ahead best forecasts over the last six months of 2007.

Results are shown in Tables 9 and 10. We report out–of–sample losses for each of the series of the SPDR and cross sectional means and quantiles across industry groups for the S&P100. The evidence from the two forecasting exercises is similar. The SPvMEM delivers the best out-of-sample performance in the majority of cases. The second best performing model is the baseline MEM(1,1) which closely tracks the SPvMEM. The performance of the MEM(2,2), on the other hand, varies substantially. In a number of cases the QL loss of this specification is much larger than the baseline MEM(1,1). The CL-MEM(1,1) does not appear to improve the predictions of the MEM(1,1). As it can be observed in Tables 5 and 6 parameters may vary quite considerably in the cross section so that, contrary to the encouraging results in GARCH panels, pooling in the MEM case may not be the best estimation strategy. The SPMEM generally performs closely to the SPvMEM but the latter generally performs better. Interestingly, in the SPDR dataset the discrepancy between the two models is larger than for the S&P100. Overall, the SPvMEM improves predictive ability in the majority of cases. An interesting outcome of the forecast comparison is that the SPvMEM delivers better results for the SPDR and for the averages of the S&P100 uniformly relative to the SPMEM. In the former model, jointly estimating the systemic risk component truly exploits the properties of interdependence across assets.

8 Conclusions

Modeling large panels of volatilities may prove a formidable task if one allows for dynamic interdependence. We follow parsimony in parametric specification, exploiting the stylized fact that volatilities appear to be driven by an underlying factor that captures the secular systematic trend. We propose a novel SeminonParametric Vector MEM (SPvMEM) specification that decomposes risk measures in a systematic and idiosyncratic components, and it allows for cross–section dependence in the innovations. The systematic component is a secular trend and the idiosyncratic components are parametric. We develop an estimation technique that is based on profile likelihood estimation and inference from the marginals. Regardless of the dimension of the panel, estimation boils down to univariate likelihood maximization and the computation of a sample correlation matrix. The ease of estimation of our model makes it appealing in large dimensional applications.

We analyse two panels of daily realized volatility measures between 2001 and 2008. The first panel consists of the nine SPDR Sectoral Indices of the S&P500, and the second panel contains the ninety constituents of the S&P100 that have been continuously trading in the sample period. There is evidence of a common trend in both panels. Once the common component is accounted for all series exhibit mean reversion around it. The model also unveils dependencies in volatility innovations across assets and sectoral clusters for Technology, Financial, Energy and Utilities companies. A forecasting horse race against a set of competing specifications shows that the SPvMEM delivers the best out–of–sample performance for the majority of series in both panels.

Further refinements or uses of the model can be envisaged. As realized measures are estimators, we would need to investigate the benefits of taking the measurement error in the volatility estimation into explicit account in the modelling step (see Hansen and Lunde (2010)). As in other contributions, a relationship between macroeconomic variables and the common component can help highlight some determinants of the changes in risk levels or, reverting the perspective, the spillover effects of market volatility onto the real economy.

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Appendix A - Assumptions

We group sets of assumptions by letter commenting hereafter on what they are needed for. Assumption A is similar to Veredas *et al.* (2007). Assumption K is standard in non–parametric estimation. Assumption B and P are standard in Maximum Likelihood estimation. Assumptions C.1, I, L, N, and S are adapted from Severini and Wong (1992) and Veredas *et al.* (2007). Assumptions C.2 and D are new to this paper.

A) Properties of the vector of innovations.

B) Existence of unique maxima.

C) Differentiability conditions (i.e. bounded derivatives) on the log–likelihoods. In particular C.2 allows for the large N setting.

D) Bounded cross–sectional covariance among innovations as $N \to \infty$.

I) Identification assumption for both the parameters and the curve.

K) Properties of the kernel.

L) Smoothness of the curve necessary to define the least favorable direction.

N) Regularity conditions for the estimated curve and its derivatives.

P) Existence of the asymptotic variance–covariance matrices of the estimated parameters.

S) Bounded mixed derivatives needed to show the convergence of the estimated tangent vector to the least favorable direction.

Assumption A

A.1 The detrended process $x_{i\,t}/\phi_0(z_t) = a_{i0}\mu_{i\,t}(\boldsymbol{\delta}_{i0})\epsilon_{i\,t}$ is a strong mixing process for any $i = 1, \dots N$, where, for some $p > 2$ and $r \in \mathbb{N}$, the mixing coefficients $\{\alpha_j\}$ must satisfy

$$\sum_{j=1}^{\infty} j^{r-1}\alpha_j^{1-2/p} < \infty.$$

Furthermore, for some even integer $q \leq 2r$

$$\mathrm{E}_0[|\mu_{i\,t}(\boldsymbol{\delta}_{i0})\epsilon_{i\,t}|^q] < m,$$

where m is a constant not depending on z_t.

A.2 $\boldsymbol{\epsilon}_t$ is a conditionally independent random vector process such that, for any $i = 1, \dots N$,

$$\epsilon_{i\,t}|\mathcal{F}_{t-1} \sim Gamma(\nu_{i0}, \nu_{i0}),$$

and $\mathrm{E}_0[\epsilon_{i\,t}] = 1$, $\mathrm{Var}_0[\epsilon_{i\,t}] = \nu_{i0}^{-1}$.

A.3 For any $i = 1, \dots, N$, we have

$$0 < \min_i(\nu_{i0}) \leq \nu_{i0} \leq \max_i(\nu_{i0}) < \infty.$$

Assumption B The true values of the parameters are such that $\boldsymbol{\xi}_0 \in int(\boldsymbol{\Xi})$ and $\boldsymbol{\psi}_0 \in int(\boldsymbol{\Psi})$, with $\boldsymbol{\Xi} \subset \mathbb{R}^{5N}$, $\boldsymbol{\Psi} \subset \mathbb{R}^{p_\psi}$ and $\boldsymbol{\Xi}$, $\boldsymbol{\Psi}$ both compact sets. We also use the notation $\boldsymbol{\eta}_0 = (\boldsymbol{\xi}_0^{\mathrm{T}}, \boldsymbol{\psi}_0^{\mathrm{T}})^{\mathrm{T}} \in int(\boldsymbol{\Lambda}) \subset \mathbb{R}^{5N+\psi}$, with $\boldsymbol{\Lambda}$ compact.

The least favorable curve is such that, for any $\boldsymbol{\xi} \in \boldsymbol{\Xi}$ and any $z_t \in [0,1]$, $\phi_{\boldsymbol{\xi}}(z_t) \in int(\mathcal{P})$, with $\mathcal{P} \subset \mathbb{R}_+$ compact.

Assumption C

C.1 We assume that for each $\boldsymbol{\xi}_i \in \boldsymbol{\Xi}_i$, $i = 1, \ldots, N$ and $z_t \in [0,1]$,

$$\sup_{\boldsymbol{\xi}_i \in \boldsymbol{\Xi}_i} \sup_{\phi \in \Gamma} \sup_{z_t \in [0,1]} \mathrm{E}_0 \left[\left| \frac{\partial^k}{\partial \boldsymbol{\xi}_i^k} \frac{\partial^l}{\partial z_t^l} \frac{\partial^j}{\partial \phi^j} \boldsymbol{\ell}_{i\,t}^m(\boldsymbol{\xi}_i, \phi(z_t)) \right|^q \right] < \infty,$$

for $j = 0,1,2,3$, $k = 0,1,2$, $l = 0,1,2$ and $q = 2$.

The same holds also for $\mathrm{E}_0\left[\boldsymbol{\ell}_t^c(\boldsymbol{\xi}, \boldsymbol{\psi}, \phi(z_t))\right]$.

C.2 We also assume that:

$$\sup_{N \in \mathbb{N}} \sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{\phi \in \Gamma} \sup_{z_t \in [0,1]} \mathrm{E}_0 \left[\left| \frac{1}{N} \sum_{i=1}^N \frac{\partial^k}{\partial \boldsymbol{\xi}^k} \frac{\partial^l}{\partial z_t^l} \frac{\partial^j}{\partial \phi^j} \boldsymbol{\ell}_{i\,t}^m(\boldsymbol{\xi}_i, \phi(z_t)) \right|^q \right] < \infty,$$

for $j = 0,1,2,3$, $k = 0,1,2$, $l = 0,1,2$, and $q = 2$.

We use the notation $\partial/\partial\phi$ to indicate the Fréchet functional derivative.

Assumption D Define $\mathrm{Cov}_0[\epsilon_{i\,t}, \epsilon_{j\,t}] = \tau_{ij}$, then there exists a positive real constant M such that

$$\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N |\tau_{ij}| < M < \infty.$$

Assumption I

I.1 For any $i = 1, \ldots, N$ and for fixed but arbitrary $\tilde{\boldsymbol{\eta}} \in \boldsymbol{\Lambda}$, $\tilde{\boldsymbol{\xi}}_i \in \boldsymbol{\Xi}_i$, $\tilde{\boldsymbol{\psi}} \in \boldsymbol{\Psi}$ and $\tilde{\phi} \in \mathcal{P}$, let

$$\rho(\boldsymbol{\eta}, \phi) = \widetilde{\mathrm{E}}\left[\boldsymbol{\ell}_t(\boldsymbol{\eta}, \phi)\right], \quad \rho_i^m(\boldsymbol{\xi}_i, \phi) = \widetilde{\mathrm{E}}\left[\boldsymbol{\ell}_{i\,t}^m(\boldsymbol{\xi}_i, \phi)\right], \quad \rho^c(\boldsymbol{\xi}, \boldsymbol{\psi}, \phi) = \widetilde{\mathrm{E}}\left[\boldsymbol{\ell}_t^c(\boldsymbol{\xi}, \boldsymbol{\psi}, \phi)\right]$$

where expectation is taken with respect to the distribution of $\mathbf{x}_t$ with parameters $\tilde{\phi}$ and $\tilde{\boldsymbol{\eta}}$, $\tilde{\boldsymbol{\xi}}_i$, or $\tilde{\boldsymbol{\psi}}$ respectively. Then, if $\phi \neq \tilde{\phi}$, we have

$$\rho(\boldsymbol{\eta}, \phi) < \rho(\tilde{\boldsymbol{\eta}}, \tilde{\phi}), \quad \rho_i^m(\boldsymbol{\xi}_i, \phi) < \rho_i^m(\tilde{\boldsymbol{\xi}}_i, \tilde{\phi}), \quad \rho^c(\boldsymbol{\xi}, \boldsymbol{\psi}, \phi) < \rho^c(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}}, \tilde{\phi}).$$

I.2 Let $\phi_{\boldsymbol{\xi}}(z_t)$ be such that

$$\frac{\partial}{\partial \phi} \mathrm{E}_0\left[\boldsymbol{\ell}_{i\,t}^m(\boldsymbol{\xi}_i, \phi_{\boldsymbol{\xi}}(z_t))\right] = 0,$$

for any $z_t \in [0,1]$ and for each fixed $\boldsymbol{\xi}_i \in \boldsymbol{\Xi}_i$, $i = 1, \ldots, N$. Then, we assume that $\phi_{\boldsymbol{\xi}}(z_t)$ is unique and that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, if

$$\sup_{\boldsymbol{\xi}_i \in \boldsymbol{\Xi}_i} \sup_{z_t \in [0,1]} \left| \frac{\partial}{\partial \phi} \mathrm{E}_0\left[\boldsymbol{\ell}_{i\,t}^m(\boldsymbol{\xi}_i, \bar{\phi}(z_t))\right] \right| \leq \delta,$$

then

$$\sup_{\boldsymbol{\xi}\in\boldsymbol{\Xi}} \sup_{z_t\in[0,1]} \left|\bar{\phi}(z_t) - \phi_{\boldsymbol{\xi}}(z_t)\right| \leq \varepsilon.$$

Assumption K Assume that the kernel function $\mathrm{K}(\cdot)$ is of order $k > 3/2$ with support $[-1,1]$ and it is such that

$$\int_{-1}^{1} \mathrm{K}(u)\mathrm{d}u = 1, \quad \int_{-1}^{1} u\mathrm{K}(u)\mathrm{d}u = 0, \quad \int_{-1}^{1} u^p\mathrm{K}(u)\mathrm{d}u < \infty, \quad \int_{-1}^{1} u^q\mathrm{K}^2(u)\mathrm{d}u < \infty,$$

for $p = 0, \ldots, 3$ and $q = 0, \ldots, 6$. Assume also that

$$\sup_{u\in[-1,1]} \left|\frac{\partial^r \mathrm{K}(u)}{\partial u^r}\right| < \infty, \qquad r = 0, \ldots, 4.$$

The conditions on the bandwidth vary and are stated in Theorem 2.

Assumption L Given the least favorable curve $\phi_{\boldsymbol{\xi}}$, then, for any $z_t \in [0,1]$ and any $\boldsymbol{\xi} \in \boldsymbol{\Xi}$, define

$$\phi'_{\boldsymbol{\xi}}(z_t) \equiv \frac{\partial \phi_{\boldsymbol{\xi}}}{\partial \boldsymbol{\xi}}(z_t) \ \text{ and } \ \phi''_{\boldsymbol{\xi}}(z_t) \equiv \frac{\partial^2 \phi_{\boldsymbol{\xi}}}{\partial \boldsymbol{\xi}^2}(z_t),$$

and define the norm of a vector $\mathbf{w}$ as

$$||\mathbf{w}|| = \sup_{z_t\in[0,1]} |\mathbf{w}(z_t)|.$$

Then, we assume that $\phi'_{\boldsymbol{\xi}}(z_t)$ and $\phi''_{\boldsymbol{\xi}}(z_t)$ exist and

$$||\phi'_{\boldsymbol{\xi}}|| < \infty \ \text{ and } \ ||\phi''_{\boldsymbol{\xi}}|| < \infty.$$

Assumption N For any $z_t \in [0,1]$ and $\boldsymbol{\xi} \in \boldsymbol{\Xi}$, the estimated curve $\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_t)$ converges in probability to some constant both if $T \to \infty$ and N is small and if both $N, T \to \infty$. Denote that constant as $\widetilde{\phi}_{\boldsymbol{\xi}}(z_t)$. For any $\boldsymbol{\xi} \in \boldsymbol{\Xi}$, we require that $\widetilde{\phi}_{\boldsymbol{\xi}} \in \Gamma$ and, and for all $r, s = 0, 1, 2$ such that $r + 2 \leq 2$, that

$$\frac{\partial^{r+s}}{\partial z_t^r \partial \boldsymbol{\xi}^s} \widetilde{\phi}_{\boldsymbol{\xi}}(z_t) \ \text{ and } \ \frac{\partial^{r+s}}{\partial z_t^r \partial \boldsymbol{\xi}^s} \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_t),$$

exist. We require that

$$\begin{aligned}
\sup_{\boldsymbol{\xi}\in\boldsymbol{\Xi}} ||\widehat{\phi}_{\boldsymbol{\xi}\,NT} - \widetilde{\phi}_{\boldsymbol{\xi}}|| &= o_P(1), \\
\sup_{\boldsymbol{\xi}\in\boldsymbol{\Xi}} ||\widehat{\phi}'_{\boldsymbol{\xi}\,NT} - \widetilde{\phi}'_{\boldsymbol{\xi}}|| &= o_P(1), \\
\sup_{\boldsymbol{\xi}\in\boldsymbol{\Xi}} ||\widehat{\phi}''_{\boldsymbol{\xi}\,NT} - \widetilde{\phi}''_{\boldsymbol{\xi}}|| &= o_P(1),
\end{aligned}$$

where the norm is defined in Assumption L.

Finally, for some $\delta > 0$, we require that

$$\begin{aligned}
\left\| \frac{\partial}{\partial z_t} \widehat{\phi}_{\boldsymbol{\xi}_o NT} - \frac{\partial}{\partial z_t} \widetilde{\phi}_0 \right\| &= o_P(T^{-\delta}), \\
\left\| \frac{\partial}{\partial z_t} \widehat{\phi}'_{\boldsymbol{\xi}_o NT} - \frac{\partial}{\partial z_t} \widetilde{\phi}'_0 \right\| &= o_P(T^{-\delta}).
\end{aligned}$$

Assumption P The following matrices are positive definite:

$$\begin{aligned}
\mathcal{I}_{\boldsymbol{\xi}_{io}\boldsymbol{\xi}_{io}} &= \mathrm{E}_0 \left[\frac{\partial \boldsymbol{\ell}_{it}^m}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i0}, \phi_0) \frac{\partial \boldsymbol{\ell}_{it}^m}{\partial \boldsymbol{\xi}_i^{\mathrm{T}}}(\boldsymbol{\xi}_{i0}, \phi_0) \right], \ \text{for } i = 1, \ldots, N, \\
\mathbf{I}_{\boldsymbol{\psi}_o} &= \mathrm{E}_0 \left[\frac{\partial \boldsymbol{\ell}_t^c}{\partial \boldsymbol{\psi}}(\boldsymbol{\xi}_0, \boldsymbol{\psi}_0, \phi_0) \frac{\partial \boldsymbol{\ell}_t^c}{\partial \boldsymbol{\psi}^{\mathrm{T}}}(\boldsymbol{\xi}_0, \boldsymbol{\psi}_0, \phi_0) \right], \\
\mathcal{H}_{\boldsymbol{\xi}_{io}\boldsymbol{\xi}_{io}} &= -\mathrm{E}_0 \left[\frac{\partial^2 \boldsymbol{\ell}_{it}^m}{\partial \boldsymbol{\xi}_i \partial \boldsymbol{\xi}_i^{\mathrm{T}}}(\boldsymbol{\xi}_{i0}, \phi_0) \right], \ \text{for } i = 1, \ldots, N, \\
\mathbf{H}_{\boldsymbol{\psi}_0} &= -\mathrm{E}_0 \left[\frac{\partial^2 \boldsymbol{\ell}_t^c}{\partial \boldsymbol{\psi} \partial \boldsymbol{\psi}^{\mathrm{T}}}(\boldsymbol{\xi}_0, \boldsymbol{\psi}_0, \phi_0) \right].
\end{aligned}$$

Moreover, the matrices $\mathbf{I}^*_{\boldsymbol{\xi}_0}$ and $\mathbf{H}^*_{\boldsymbol{\xi}_o}$ defined in Theorem 4 are positive definite.

We also assume

$$\bar{\jmath}_{N\boldsymbol{\xi}_o}(z_\tau) = -\frac{1}{N} \mathrm{E}_0 \left[\sum_{i=1}^N \frac{\partial^2 \boldsymbol{\ell}_{it}^m}{\partial \phi^2}(\boldsymbol{\xi}_{i0}, \phi_0(z_\tau)) \right] > 0, \ z_\tau \in [0, 1].$$

Assumption S Assume that for all $r, s = 0, \ldots, 4$, $r+s \leq 4$, and any $i = 1, \ldots, N$, the derivative

$$\frac{\partial^{r+s} \boldsymbol{\ell}_{it}^m}{\partial \boldsymbol{\xi}_i^r \partial \phi^s}(\boldsymbol{\xi}_i, \phi)$$

exist for almost all $\mathbf{x}_t$ and assume that

$$\mathrm{E}_0 \left[\sup_{\boldsymbol{\eta} \in \Lambda} \sup_{\phi \in \Gamma} \left\| \frac{\partial^{r+s} \boldsymbol{\ell}_{it}^m}{\partial \boldsymbol{\xi}_i^r \partial \phi^s}(\boldsymbol{\xi}_i, \phi) \frac{\partial^{r+s} \boldsymbol{\ell}_{it}^m}{\partial \boldsymbol{\xi}_i^{r\mathrm{T}} \partial \phi^s}(\boldsymbol{\xi}_i, \phi) \right\| \right] < \infty.$$

Appendix B - Proofs of Theorems

Additional results

We need four Lemma to prove the Theorems and Corollary 1. Notice that, although not explicitly indicated in the proofs, all density functions f_{x_i} and c are to be considered as conditional on $\mathcal{F}_{t-1}$.

Lemma 1 *Under assumptions C.1 and S*

$$\begin{aligned}
&\mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}_{i\,t}^{m}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i\,0},\phi_0)\right]=\mathbf{0},\ \textit{for}\ \ i=1,\dots,N,\\
&\mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}_{t}^{c}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0,\phi_0)\right]=\mathbf{0},\ \textit{for}\ \ i=1,\dots,N,\\
&\mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}_{t}^{c}}{\partial \boldsymbol{\psi}}(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0,\phi_0)\right]=\mathbf{0},\\
&\mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}_{i\,t}^{m}}{\partial \phi}(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right]=0,\ \textit{for}\ \ i=1,\dots,N,\ z_\tau\in[0,1],\\
&\mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}_{t}^{c}}{\partial \phi}(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0,\phi_0(z_\tau))\right]=0,\ \textit{for}\ \ z_\tau\in[0,1].
\end{aligned}$$

Proof. We prove just the first relation, the proof of the others being analogous:

$$\begin{aligned}
&\mathrm{E}_0\left[\frac{\partial}{\partial \boldsymbol{\xi}_i}\boldsymbol{\ell}_{i\,t}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0)\right]=\\
=\ &\int_{\mathbf{x}}\left(\frac{\partial}{\partial \boldsymbol{\xi}_i}\log f_{x_i}(\boldsymbol{\xi}_{i\,0},\phi_0)\right)\left(\prod_{k=1}^{N} f_{x_k}(\boldsymbol{\xi}_{k\,0},\phi_0)\right)c(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0,\phi_0)\mathrm{d}\mathbf{x}_t=\\
=\ &\int_{\mathbf{x}}\left(\frac{\partial}{\partial \boldsymbol{\xi}_i}f_{x_i}(\boldsymbol{\xi}_{i\,0},\phi_0)\right)\frac{\left(\prod_{k=1}^{N} f_{x_k}(\boldsymbol{\xi}_{k\,0},\phi_0)\right)}{f_{x_i}(\boldsymbol{\xi}_{i\,0},\phi_0)}c(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0,\phi_0)\mathrm{d}\mathbf{x}_t=\\
=\ &\frac{\partial}{\partial \boldsymbol{\xi}_i}\left(\int_{\mathbf{x}}\left(\prod_{k=1}^{N} f_{x_k}(\boldsymbol{\xi}_{k\,0},\phi_0)\right)c(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0,\phi_0)\mathrm{d}\mathbf{x}_t\right)=\frac{\partial}{\partial \boldsymbol{\xi}_i}1=0.\ \square
\end{aligned}$$

Lemma 2 *Under assumptions C.1, S and P, for any* $i,j=1,\dots,N$

$$\mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}_{i\,t}^{m}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i\,0},\phi_0)\frac{\partial \boldsymbol{\ell}_{t}^{c}}{\partial \boldsymbol{\psi}^{\intercal}}(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0,\phi_0)\right]=\mathbf{0}, \tag{B-1}$$

and

$$\mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}_{i\,t}^{m}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i\,0},\phi_0)\frac{\partial \boldsymbol{\ell}_{j\,t}^{m}}{\partial \boldsymbol{\xi}_j^{\intercal}}(\boldsymbol{\xi}_{j\,0},\phi_0)\right]=\mathbf{0}. \tag{B-2}$$

Proof. Equation (B-1) is in the appendix in Joe (2005). The proof of (B-2) is similar. Let $\mathbf{x}_{-i\,t}$ be the vector $\mathbf{x}_t$ when omitting the i-th component. We omit the dependence on ϕ_0 for simplicity. Then the expectation in (B-2) is equivalent to

$$
\begin{aligned}
&\mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}_{it}^m}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i\,0})\frac{\partial \boldsymbol{\ell}_{jt}^m}{\partial \boldsymbol{\xi}_j^{\mathrm{T}}}(\boldsymbol{\xi}_{j\,0})\right] = \\
= \;& \int_{\mathbf{x}}\left(\frac{\partial}{\partial \boldsymbol{\xi}_i}\log f_{x_i}(\boldsymbol{\xi}_{i\,0})\right)\left(\frac{\partial}{\partial \boldsymbol{\xi}_j^{\mathrm{T}}}\log f_{x_j}(\boldsymbol{\xi}_{j\,0})\right)\left(\prod_{k=1}^{N} f_{x_k}(\boldsymbol{\xi}_{k\,0})\right)c(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0)\mathrm{d}\mathbf{x}_t = \\
= \;& \int_{x_i}\left(\frac{\partial}{\partial \boldsymbol{\xi}_i}\log f_{x_i}(\boldsymbol{\xi}_{i\,0})\right)\left[\int_{\mathbf{x}_{-i}}\left(\frac{\partial}{\partial \boldsymbol{\xi}_j^{\mathrm{T}}}\log f_{x_j}(\boldsymbol{\xi}_{j\,0})\right)\left(\prod_{k=1}^{N} f_{x_k}(\boldsymbol{\xi}_{k\,0})\right)c(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0)\mathrm{d}\mathbf{x}_{-i\,t}\right]\mathrm{d}x_{i\,t} = \\
= \;& \int_{x_i}\left(\frac{\partial}{\partial \boldsymbol{\xi}_i}\log f_{x_i}(\boldsymbol{\xi}_{i\,0})\right)\left[\int_{\mathbf{x}_{-i}}\left(\frac{\partial}{\partial \boldsymbol{\xi}_j^{\mathrm{T}}} f_{x_j}(\boldsymbol{\xi}_{j\,0})\right)\left(\prod_{k\neq j;k=1}^{N} f_{x_k}(\boldsymbol{\xi}_{k\,0})\right)c(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0)\mathrm{d}\mathbf{x}_{-i\,t}\right]\mathrm{d}x_{i\,t} = \\
= \;& \int_{x_i}\left(\frac{\partial}{\partial \boldsymbol{\xi}_i}\log f_{x_i}(\boldsymbol{\xi}_{i\,0})\right)\frac{\partial}{\partial \boldsymbol{\xi}_j^{\mathrm{T}}}\left[\int_{\mathbf{x}_{-i}}\left(\prod_{k=1}^{N} f_{x_k}(\boldsymbol{\xi}_{k\,0})\right)c(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0)\mathrm{d}\mathbf{x}_{-i\,t}\right]\mathrm{d}x_{i\,t} = \\
= \;& \int_{x_i}\left(\frac{\partial}{\partial \boldsymbol{\xi}_i}\log f_{x_i}(\boldsymbol{\xi}_{i\,0})\right)\frac{\partial}{\partial \boldsymbol{\xi}_j^{\mathrm{T}}} f_{x_i}(\boldsymbol{\xi}_{i\,0})\mathrm{d}x_{i\,t} = \mathbf{0}. \;\square
\end{aligned}
$$

Lemma 3 *Under assumptions C.1, N and S*

a) for any $z_t \in [0,1]$, *as* $T \to \infty$,

$$
\begin{aligned}
&\left\|\frac{1}{\sqrt{T}}\frac{\partial}{\partial \boldsymbol{\xi}_i}\frac{\partial \sum_{t=1}^{T}\boldsymbol{\ell}_{it}^m}{\partial \phi}(\boldsymbol{\xi}_{i\,0},\phi_0)(\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_t)-\phi_0(z_t))\right\|_2 = o_P(1), \;\; \textit{for} \;\; i=1,\ldots,N, \\
&\left\|\frac{1}{\sqrt{T}}\frac{\partial}{\partial \boldsymbol{\psi}}\frac{\partial \sum_{t=1}^{T}\boldsymbol{\ell}_{t}^c}{\partial \phi}(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0,\phi_0)(\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_t)-\phi_0(z_t))\right\|_2 = o_P(1).
\end{aligned}
$$

b) for any $z_t \in [0,1]$, *as* $T \to \infty$,

$$
\begin{aligned}
&\left\|\frac{1}{\sqrt{T}}\frac{\partial \sum_{t=1}^{T}\boldsymbol{\ell}_{it}^m}{\partial \phi}(\boldsymbol{\xi}_{i\,0},\phi_0)(\widehat{\phi}'_{\boldsymbol{\xi}_o\,NT}(z_t)-\phi'_0(z_t))\right\|_2 = o_P(1), \;\; \textit{for} \;\; i=1,\ldots,N, \\
&\left\|\frac{1}{\sqrt{T}}\frac{\partial \sum_{t=1}^{T}\boldsymbol{\ell}_{t}^c}{\partial \phi}(\boldsymbol{\xi}_0,\boldsymbol{\psi}_0,\phi_0)(\widehat{\phi}'_{\boldsymbol{\xi}_o\,NT}(z_t)-\phi'_0(z_t))\right\|_2 = o_P(1).
\end{aligned}
$$

c) for any $\boldsymbol{\xi}_i \in \boldsymbol{\Xi}_i$ *and* $\boldsymbol{\psi} \in \boldsymbol{\Psi}$,

$$
\begin{aligned}
&\sum_{t=1}^{T}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_i,\widehat{\phi}_{\boldsymbol{\xi}\,T}) - \sum_{t=1}^{T}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_i,\phi_{\boldsymbol{\xi}}) = r^{(1)}(\boldsymbol{\xi}_i), \;\; \textit{for} \;\; i=1,\ldots,N, \\
&\sum_{t=1}^{T}\boldsymbol{\ell}_{t}^c(\boldsymbol{\xi},\boldsymbol{\psi},\widehat{\phi}_{\boldsymbol{\xi}\,T}) - \sum_{t=1}^{T}\boldsymbol{\ell}_{t}^c(\boldsymbol{\xi},\boldsymbol{\psi},\phi_{\boldsymbol{\xi}}) = r^{(3)}(\boldsymbol{\psi}),
\end{aligned}
$$

such that, as $T \to \infty$,

$$\sup_{\boldsymbol{\xi}_i \in \boldsymbol{\Xi}_i} \left|\left| \frac{1}{T} \frac{\partial^2 r^{(1)}}{\partial \boldsymbol{\xi}_i \partial \boldsymbol{\xi}_i^{\tau}}(\boldsymbol{\xi}_i) \right|\right|_2 = o_P(1), \ \ \textit{for} \ \ i = 1, \ldots, N$$
$$\sup_{\boldsymbol{\psi} \in \boldsymbol{\Psi}} \left|\left| \frac{1}{T} \frac{\partial^2 r^{(3)}}{\partial \boldsymbol{\psi} \partial \boldsymbol{\psi}^{\tau}}(\boldsymbol{\psi}) \right|\right|_2 = o_P(1).$$

d) for any $\boldsymbol{\xi}_i \in \boldsymbol{\Xi}_i$ and $\boldsymbol{\psi} \in \boldsymbol{\Psi}$,

$$\begin{aligned}
\sum_{t=1}^{T} \boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_i, \widehat{\phi}_{\boldsymbol{\xi}\,T}) &= \sum_{t=1}^{T} \boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_i, \phi_{\boldsymbol{\xi}}) + \\
&\quad + \frac{\partial}{\partial \phi} \sum_{t=1}^{T} \boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_i, \phi_{\boldsymbol{\xi}})(\widehat{\phi}_{\boldsymbol{\xi}}(z_t) - \phi_{\boldsymbol{\xi}}(z_t)) + \\
&\quad + r^{(2)}(\boldsymbol{\xi}_i), \ \ \textit{for} \ \ i = 1, \ldots, N, \\
\sum_{t=1}^{T} \boldsymbol{\ell}_{t}^{c}(\boldsymbol{\xi}, \boldsymbol{\psi}, \widehat{\phi}_{\boldsymbol{\xi}\,T}) &= \sum_{t=1}^{T} \boldsymbol{\ell}_{t}^{c}(\boldsymbol{\xi}, \boldsymbol{\psi}, \phi_{\boldsymbol{\xi}}) + \\
&\quad + \frac{\partial}{\partial \phi} \sum_{t=1}^{T} \boldsymbol{\ell}_{t}^{c}(\boldsymbol{\xi}, \boldsymbol{\psi}, \phi_{\boldsymbol{\xi}})(\widehat{\phi}_{\boldsymbol{\xi}}(z_t) - \phi_{\boldsymbol{\xi}}(z_t)) + \\
&\quad + r^{(4)}(\boldsymbol{\psi}),
\end{aligned}$$

such that, as $T \to \infty$,

$$\left|\left| \frac{1}{\sqrt{T}} \frac{\partial r^{(2)}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i\,0}) \right|\right|_2 = o_P(1), \ \ \textit{for} \ \ i = 1, \ldots, N$$
$$\left|\left| \frac{1}{\sqrt{T}} \frac{\partial r^{(4)}}{\partial \boldsymbol{\psi}}(\boldsymbol{\psi}_0) \right|\right|_2 = o_P(1).$$

Proof. The proof is in Lemma 2 and 3 by Severini and Wong (1992) and we use the Central Limit Theorem 2.11 in Wooldridge and White (1988).

Lemma 4 *Under assumptions C.1 and K, assumption N is satisfied with $\widetilde{\phi}_{\boldsymbol{\xi}} = \phi_{\boldsymbol{\xi}}$, i.e., as $T \to \infty$, $Th_{NT} \to \infty$, $h_{NT} \to 0$,*

$$\sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} ||\widehat{\phi}_{\boldsymbol{\xi}\,NT} - \phi_{\boldsymbol{\xi}}|| = \sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{z_\tau \in [0,1]} |\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) - \phi_{\boldsymbol{\xi}}(z_\tau)| = o_P(1), \tag{B-3}$$
$$\sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} ||\widehat{\phi}'_{\boldsymbol{\xi}\,NT} - \phi'_{\boldsymbol{\xi}}|| = \sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{z_\tau \in [0,1]} |\widehat{\phi}'_{\boldsymbol{\xi}\,NT}(z_\tau) - \phi'_{\boldsymbol{\xi}}(z_\tau)| = o_P(1), \tag{B-4}$$
$$\sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} ||\widehat{\phi}''_{\boldsymbol{\xi}\,NT} - \phi''_{\boldsymbol{\xi}}|| = \sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{z_\tau \in [0,1]} |\widehat{\phi}''_{\boldsymbol{\xi}\,NT}(z_\tau) - \phi''_{\boldsymbol{\xi}}(z_\tau)| = o_P(1), \tag{B-5}$$

where $\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)$ and $\phi_{\boldsymbol{\xi}}(z_\tau)$ are defined in (B-73) *and* (B-75) *respectively. If also assumption C.2 holds then the same results are valid also if $N \to \infty$.*

Proof. We define

$$\bar{\lambda}_N(\boldsymbol{\xi}, \phi(z_\tau)) = \frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N} \boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_i, \phi(z_\tau))\right], \tag{B-6}$$

and

$$\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \phi(z_\tau)) = \sum_{t=1}^{T}\sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_i, \phi(z_t)). \tag{B-7}$$

Then define $R(z_t) = \phi_0(z_t)/\phi(z_t)$. We then have

$$\begin{aligned}
\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \phi(z_\tau)) &= \sum_{t=1}^{T}\sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_i, \phi_0(z_t)R(z_t)) = \\
&= \sum_{t=1}^{T}\sum_{i=1}^{N} \mathrm{K}(u_t)\boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_i, \phi_0(z_t)(R(z_\tau) + R'(z_\tau)h_{NT}u_t)) = \\
&= \sum_{t=1}^{T}\sum_{i=1}^{N} \mathrm{K}(u_t)\boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_i, \phi_0(z_t)R(z_\tau)) + \\
&\quad + \sum_{t=1}^{T}\sum_{i=1}^{N} \mathrm{K}(u_t)R'(z_\tau)h_{NT}u_t \frac{\partial}{\partial R}\boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_i, \phi_0(z_t)R(z_\tau)) = \\
&= A_{NT} + B_{NT}.
\end{aligned} \tag{B-8}$$

Notice that A_{NT} can be written as

$$A_{NT} = \widetilde{A}_{NT} + O_P(T^{-1}), \tag{B-9}$$

where $\widetilde{A}_{NT}$ is a sum of mixing random variables. To see this consider for simplicity the case $\gamma_i = 0$ and notice that

$$\mu_{i\,t}(\boldsymbol{\delta}_i) = \omega_i + \alpha_i \frac{x_{i\,t-1}}{a_i\phi(z_{t-1})} + \beta_i\mu_{i\,t-1} = \frac{\omega_i}{1-\beta_i} + \sum_{k=1}^{t}\alpha_i\beta_i^{k-1}\frac{x_{i\,t-k}}{a_i\phi(z_{t-k})} \tag{B-10}$$

where $\omega_i = 1 - \alpha_i - \beta_i$. We then have

$$\begin{aligned}
\mu_{i\,t}(\boldsymbol{\delta}_i) &= \frac{\omega_i}{1-\beta_i} + \sum_{k=1}^{t}\alpha_i\beta_i^{k-1}\frac{x_{i\,t-k}}{a_i\phi_0(z_{t-k})}R(z_{t-k}) = \\
&= \left[\frac{\omega_i}{1-\beta_i} + \sum_{k=1}^{t}\alpha_i\beta_i^{k-1}\frac{x_{i\,t-k}}{a_i\phi_0(z_{t-k})}R(z_\tau)\right] + \\
&\quad + \sum_{k=1}^{t}\alpha_i\beta_i^{k-1}\frac{x_{i\,t-k}}{a_i\phi_0(z_{t-k})}R'(z_{t-\bar{k}})(z_{t-k} - z_\tau) = \\
&= \chi_{1i\,t} + \chi_{2i\,t},
\end{aligned} \tag{B-11}$$

where $z_{t-k} < z_{t-\bar{k}} < z_t$, i.e. $0 \le \bar{k} \le k$. Then,

$$\chi_{2i\,t} = \sum_{k=1}^{t}\alpha_i\beta_i^{k-1}\frac{x_{i\,t-k}}{a_i\phi_0(z_{t-k})}R'(z_{t-\bar{k}})(z_{t-k} - z_\tau) = \frac{1}{T}\sum_{k=1}^{t}\alpha_i\beta_i^{k-1}\frac{x_{i\,t-k}}{a_i\phi_0(z_{t-k})}R'(z_{t-\bar{k}})\left(t - k - \tau\right).$$

Notice that $\mathrm{E}_0[\chi_{2i\,t}] = O(T^{-1})$ and

$$
\begin{aligned}
\mathrm{E}_0[\chi^2_{2i\,t}] &= \frac{1}{T^2}\mathrm{E}_0\left[\left(\sum_{k=1}^{t}\alpha_i\beta_i^{k-1}\frac{x_{i\,t-k}}{a_i\phi_0(z_{t-k})}R'(z_{t-\bar{k}})\,(t-k-\tau)\right)^2\right] \leq \\
&\leq \frac{1}{T}\mathrm{E}_0\left[\frac{1}{T}\left\{\sum_{k=1}^{t}\left(\frac{x_{i\,t-k}}{a_i\phi_0(z_{t-k})}\right)^2\right\}\left\{\sum_{k=1}^{t}(\alpha_i\beta_i^{k-1}R'(z_{t-\bar{k}})\,(t-k-\tau))^2\right\}\right] \leq \\
&\leq O\left(\frac{1}{T}\right).
\end{aligned}
$$

Thus $\mathrm{Var}_0[\chi_{2i\,t}] = O(T^{-1})$, which implies $\chi_{2i\,t} = O_P(T^{-1})$.

By using a Taylor approximation of A_{NT} we obtain (B-9). Therefore, each term of the sum in $\widetilde{A}_{NT}$ is the log–likelihood with the generic trend $\phi(z_t)$ replaced by $\phi_0(z_t)R(z_\tau)$ and $\mu_{i\,t}$ replaced by $\chi_{1i\,t}$. Notice that with this substitution $\widetilde{A}_{NT}$ is just function of the process

$$
\widetilde{x}_{i\,t} = \frac{x_{i\,t}}{a_i\phi_0(z_t)} = \frac{a_{i\,0}\epsilon_{i\,t}\mu_{i\,t}(\boldsymbol{\delta}_{i0})}{a_i},
$$

which is mixing by assumption A.1, and of $R(z_\tau)$ which for a fixed z_τ can be treated as a constant. By using the Weak Law of Large Numbers by McLeish (1975) and Lemma 2.1 of White and Domowitz (1984) for $\widetilde{A}_{NT}$ and using (B-9), we have, as $T\to\infty$ and $Th_{NT}\to\infty$,

$$
\frac{A_{NT}}{NTh_{NT}} \xrightarrow{P} \mathrm{E}_0\left[\frac{A_{NT}}{NTh_{NT}}\right]. \tag{B-12}
$$

Now let us compute

$$
\begin{aligned}
\mathrm{E}_0\left[\frac{A_{NT}}{NTh_{NT}}\right] &= \frac{1}{NTh_{NT}}\mathrm{E}_0\left[\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}(u_t)\boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i,\phi_0(z_t)R(z_\tau))\right] = \\
&= \frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{K}(u_t)\mathrm{E}_0\left[\sum_{i=1}^{N}\boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i,\phi_0(z_\tau+h_{NT}u_t)R(z_\tau))\right] = \\
&= \frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{K}(u_t)\mathrm{E}_0\left[\sum_{i=1}^{N}\boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i,\phi_0(z_\tau)R(z_\tau))\right] + \\
&\quad+\frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{K}(u_t)h_{NT}u_t\phi_0'(z_\tau)\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i,\phi_0(z_\tau)R(z_\tau))\right].
\end{aligned}
$$

Notice that $\phi_0(z_\tau)R(z_\tau) = \phi(z_\tau)$. When $T\to\infty$ and $Th_{NT}\to\infty$ and using expressions analogous to (B-33) in the proof of Theorem 2 we have

$$
\mathrm{E}_0\left[\frac{A_{NT}}{NTh_{NT}}\right] \to \frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N}\boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i,\phi(z_\tau))\right] = \bar{\lambda}_N(\boldsymbol{\xi},\phi(z_\tau)).
$$

Then, it is possible to show that $\mathrm{Var}_0[B_{NT}/NTh_{NT}] = O(h^2_{NT})$. Therefore, as $h_{NT}\to 0$ and by assumption C.1, $B_{NT}/NTh_{NT} \xrightarrow{P} \mathrm{E}_0[B_{NT}/NTh_{NT}]$. But, as $T\to\infty$ and

$Th_{NT} \to \infty$ and using expressions analogous to (B-33) in the proof of Theorem 2 we have

$$\mathrm{E}_0 \left[\frac{B_{NT}}{NTh_{NT}} \right] \to 0.$$

Therefore, we have, for any $\boldsymbol{\xi} \in \boldsymbol{\Xi}$, any $z_\tau \in [0,1]$, and any $\phi \in \mathcal{P}$,

$$\frac{\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \phi(z_\tau))}{NTh_{NT}} \xrightarrow{P} \bar{\lambda}_N(\boldsymbol{\xi}, \phi(z_\tau)), \text{ as } T \to \infty, Th_{NT} \to \infty, h_{NT} \to 0.$$

Following the same reasoning as in the proof of Lemma 8 by Severini and Wong (1992), we can also prove that, as $T \to \infty$, $Th_{NT} \to \infty$, and $h_{NT} \to 0$,

$$\sup_{\boldsymbol{\xi}_i \in \boldsymbol{\Xi}_i} \sup_{\phi \in \Gamma} \sup_{z_\tau \in [0,1]} \left| \frac{\partial^k}{\partial \boldsymbol{\xi}_i^k} \frac{\partial^l}{\partial z_t^l} \frac{\partial^j}{\partial \phi^j} \left(\frac{\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \phi(z_\tau))}{NTh_{NT}} - \bar{\lambda}_N(\boldsymbol{\xi}, \phi(z_\tau)) \right) \right| = o_P(1), \quad \text{(B-13)}$$

for $k, l, j = 0, 1, 2$.

Then, $\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)$ is such that

$$\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) = \arg\sup_{\phi \in \Gamma} \widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \phi(z_\tau)), \quad \text{(B-14)}$$

which implies

$$\frac{1}{NTh_{NT}} \frac{\partial \widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau))}{\partial \phi} = 0. \quad \text{(B-15)}$$

While, $\phi_{\boldsymbol{\xi}}(z_\tau)$ is such that, for N fixed,

$$\phi_{\boldsymbol{\xi}}(z_\tau) = \arg\sup_{\phi \in \Gamma} \bar{\lambda}_N(\boldsymbol{\xi}, \phi(z_\tau)), \quad \text{(B-16)}$$

which implies

$$\frac{\partial \bar{\lambda}_N(\boldsymbol{\xi}, \phi_{\boldsymbol{\xi}}(z_\tau))}{\partial \phi} = 0. \quad \text{(B-17)}$$

First, let us prove (B-3). For any $\varepsilon > 0$, there exists $\delta > 0$ such that, as $T \to \infty$, $Th_{NT} \to \infty$, and $h_{NT} \to 0$,

$$\begin{aligned}
&P\left\{ \sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{z_\tau \in [0,1]} \left| \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) - \phi_{\boldsymbol{\xi}}(z_\tau) \right| > \varepsilon \right\} \leq \\
&\leq P\left\{ \sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{z_\tau \in [0,1]} \left| \frac{\partial \bar{\lambda}_N(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau))}{\partial \phi} \right| > \delta \right\} = \\
&= P\left\{ \sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{z_\tau \in [0,1]} \left| \frac{1}{NTh_{NT}} \frac{\partial \widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau))}{\partial \phi} - \frac{\partial \bar{\lambda}_N(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau))}{\partial \phi} \right| > \delta \right\} \leq \\
&\leq P\left\{ \sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{z_\tau \in [0,1]} \sup_{\phi \in \Gamma} \left| \frac{1}{NTh_{NT}} \frac{\partial \widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \phi(z_\tau))}{\partial \phi} - \frac{\partial \bar{\lambda}_N(\boldsymbol{\xi}, \phi(z_\tau))}{\partial \phi} \right| > \delta \right\} \to 0,
\end{aligned}$$

where we used assumption I.2, (B-15), and (B-13). Hence,

$$\sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{z_\tau \in [0,1]} |\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) - \phi_{\boldsymbol{\xi}}(z_\tau)| = o_P(1),$$

which implies that, for any $\boldsymbol{\xi} \in \boldsymbol{\Xi}$ and any $z_\tau \in [0,1]$,

$$\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) \xrightarrow{P} \phi_{\boldsymbol{\xi}}(z_\tau), \text{ as } T \to \infty,\, Th_{NT} \to \infty,\, h_{NT} \to 0. \qquad \text{(B-18)}$$

Then, we prove (B-4). From (B-15) and (B-17), we have

$$\begin{aligned}
0 &= \frac{1}{NTh_{NT}} \frac{\partial \widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau))}{\partial \phi} - \frac{\partial \bar{\lambda}_N(\boldsymbol{\xi}, \phi_{\boldsymbol{\xi}}(z_\tau))}{\partial \phi} = \\
&= \frac{1}{NTh_{NT}} \frac{\partial \widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau))}{\partial \phi} - \frac{\partial \bar{\lambda}_N(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau))}{\partial \phi} + \frac{\partial \bar{\lambda}_N(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau))}{\partial \phi} - \frac{\partial \bar{\lambda}_N(\boldsymbol{\xi}, \phi_{\boldsymbol{\xi}}(z_\tau))}{\partial \phi} = \\
&= R_{NT}(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)) + \Delta_N(\boldsymbol{\xi}, \bar{\phi}(z_\tau)) \left(\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) - \phi_{\boldsymbol{\xi}}(z_\tau)\right), \qquad \text{(B-19)}
\end{aligned}$$

where $\bar{\phi}(z_\tau)$ lies between $\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)$ and $\phi_{\boldsymbol{\xi}}(z_\tau)$, and

$$\begin{aligned}
R_{NT}(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)) &= \frac{1}{NTh_{NT}} \frac{\partial \widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau))}{\partial \phi} - \frac{\partial \bar{\lambda}_N(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau))}{\partial \phi}, \\
\Delta_N(\boldsymbol{\xi}, \bar{\phi}(z_\tau)) &= \frac{\partial^2 \bar{\lambda}_N(\boldsymbol{\xi}, \bar{\phi}(z_\tau))}{\partial \phi^2}.
\end{aligned}$$

By differentiating (B-19) with respect to $\boldsymbol{\xi}$, we have

$$\begin{aligned}
\mathbf{0} &= \frac{\partial}{\partial \boldsymbol{\xi}} R_{NT}(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)) + \frac{\partial}{\partial \boldsymbol{\xi}} \Delta_N(\boldsymbol{\xi}, \bar{\phi}(z_\tau)) \left(\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) - \phi_{\boldsymbol{\xi}}(z_\tau)\right) + \\
&\quad + \Delta_N(\boldsymbol{\xi}, \bar{\phi}(z_\tau)) \left(\widehat{\phi}'_{\boldsymbol{\xi}\,NT}(z_\tau) - \phi'_{\boldsymbol{\xi}}(z_\tau)\right). \qquad \text{(B-20)}
\end{aligned}$$

From assumption C.1 we have

$$\sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{z_\tau \in [0,1]} \left| \frac{\partial}{\partial \boldsymbol{\xi}} \Delta_N(\boldsymbol{\xi}, \bar{\phi}(z_\tau)) \right| = O_P(1),$$

and from (B-13) we have

$$\sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{z_\tau \in [0,1]} \left| \frac{\partial}{\partial \boldsymbol{\xi}} R_{NT}(\boldsymbol{\xi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)) \right| = o_P(1).$$

Therefore, from (B-20), we have

$$\sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \sup_{z_\tau \in [0,1]} |\widehat{\phi}'_{\boldsymbol{\xi}\,NT}(z_\tau) - \phi'_{\boldsymbol{\xi}}(z_\tau)| = o_P(1),$$

which implies that, for any $\boldsymbol{\xi} \in \boldsymbol{\Xi}$ and any $z_\tau \in [0,1]$,

$$\widehat{\phi}'_{\boldsymbol{\xi}\,NT}(z_\tau) \xrightarrow{P} \phi'_{\boldsymbol{\xi}}(z_\tau), \text{ as } T \to \infty,\, Th_{NT} \to \infty,\, h_{NT} \to 0. \qquad \text{(B-21)}$$

Finally, as $N, T \to \infty$, $Th_{NT} \to \infty$, $h_{NT} \to 0$, (B-13) becomes

$$\sup_{\boldsymbol{\xi} \in \boldsymbol{\Xi}_i} \sup_{\phi \in \Gamma} \sup_{z_t \in [0,1]} \left| \frac{\partial^k}{\partial \boldsymbol{\xi}_i^k} \frac{\partial^l}{\partial z_t^l} \frac{\partial^j}{\partial \phi^j} \left(\frac{\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}, \phi(z_\tau))}{NTh_{NT}} - \lambda(\boldsymbol{\xi}, \phi(z_\tau)) \right) \right| = o_P(1),$$

where $\lambda(\boldsymbol{\xi}, \phi(z_\tau)) = \lim_{N \to \infty} \bar{\lambda}_N(\boldsymbol{\xi}, \phi(z_\tau))$ which exists by assumption C.2. Using this assumption (B-3) can be proved as well. Similarly, (B-4) can be proved by taking the limit for $N \to \infty$ in (B-19) and (B-20). All limits exist by virtue of assumption C.2. Using the same approach we can prove also (B-5). $\square$

Proof of Theorem 1

a) Define for any $z_\tau \in [0,1]$ the smoothed likelihood

$$\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}_0, \phi(z_\tau)) \equiv \sum_{t=1}^{T}\sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0}, \phi(z_t)). \tag{B-22}$$

Then the estimated curve is such that

$$\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau) = \arg\sup_{\phi\in\Gamma} \widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}_0, \phi(z_\tau)), \tag{B-23}$$

and, from Lemma 1 and assumption P, the true value of the curve for N fixed is such that it maximizes the marginal log–likelihoods:

$$\phi_0(z_\tau) = \arg\sup_{\phi\in\Gamma} \bar{\lambda}_N(\boldsymbol{\xi}_0, \phi(z_\tau)), \tag{B-24}$$

where

$$\bar{\lambda}_N(\boldsymbol{\xi}_0, \phi(z_\tau)) = \frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N} \boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0}, \phi(z_\tau))\right]. \tag{B-25}$$

Then, following the same argument as in the proof of Lemma 4, we can use the Weak Law of Large Numbers by McLeish (1975) and Lemma 2.1 of White and Domowitz (1984). Thus, we have, for any $z_\tau \in [0,1]$ and $\phi \in \mathcal{P}$,

$$\frac{\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}_0, \phi(z_\tau))}{NTh_{NT}} \xrightarrow{P} \bar{\lambda}_N(\boldsymbol{\xi}_0, \phi(z_\tau)), \text{ as } T\to\infty,\, Th_{NT}\to\infty,\, h_{NT}\to 0. \tag{B-26}$$

Furthermore, since, for any $z_\tau \in [0,1]$,

$$\sup_{\phi\in\Gamma}\frac{\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}_0, \phi(z_\tau))}{NTh_{NT}} \xrightarrow{P} \sup_{\phi\in\Gamma}\bar{\lambda}_N(\boldsymbol{\xi}_0, \phi(z_\tau)), \text{ as } T\to\infty,\, Th_{NT}\to\infty,\, h_{NT}\to 0,$$

given (B-23) and (B-24), we have, for any $z_\tau \in [0,1]$,

$$\frac{\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}_0, \widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau))}{NTh_{NT}} \xrightarrow{P} \bar{\lambda}_N(\boldsymbol{\xi}_0, \phi_0(z_\tau)), \text{ as } T\to\infty,\, Th_{NT}\to\infty,\, h_{NT}\to 0. \tag{B-27}$$

By applying (B-26) to the left hand side of (B-27), we have, for any $z_\tau \in [0,1]$,

$$\bar{\lambda}_N(\boldsymbol{\xi}_0, \widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau)) \xrightarrow{P} \bar{\lambda}_N(\boldsymbol{\xi}_0, \phi_0(z_\tau)), \text{ as } T\to\infty,\, Th_{NT}\to\infty,\, h_{NT}\to 0.$$

Assumptions I and S imply that, for any $z_\tau \in [0,1]$, and for any N fixed, we have

$$\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau) \xrightarrow{P} \phi_0(z_\tau), \text{ as } T\to\infty,\, Th_{NT}\to\infty,\, h_{NT}\to 0.$$

b) Now let us consider the case $N \to \infty$. By assumption C.2 we know that

$$\sup_{N \in \mathbb{N}} \left| \bar{\lambda}_N(\boldsymbol{\xi}_0, \phi(z_\tau)) \right| < \infty.$$

Therefore, the following limit exists, and we define

$$\lambda(\boldsymbol{\xi}_0, \phi(z_\tau)) = \lim_{N \to \infty} \bar{\lambda}_N(\boldsymbol{\xi}_0, \phi(z_\tau)).$$

Summing up we have that for any $N \in \mathbb{N}$

$$\phi_0(z_\tau) = \arg\sup_{\phi \in \Gamma} \lambda(\boldsymbol{\xi}_0, \phi(z_\tau)). \tag{B-28}$$

Moreover,

$$\frac{\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}_0, \phi(z_\tau))}{NTh_{NT}} \xrightarrow{P} \lambda(\boldsymbol{\xi}_0, \phi(z_\tau)), \text{ as } N, T \to \infty, Th_{NT} \to \infty, h_{NT} \to 0. \tag{B-29}$$

By the same arguments as before we have, for any $z_\tau \in [0, 1]$,

$$\frac{\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}_0, \widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_\tau))}{NTh_{NT}} \xrightarrow{P} \lambda(\boldsymbol{\xi}_0, \phi_0(z_\tau)), \text{ as } N, T \to \infty, Th_{NT} \to \infty, h_{NT} \to 0,$$

and

$$\lambda(\boldsymbol{\xi}_0, \widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_\tau)) \xrightarrow{P} \lambda(\boldsymbol{\xi}_0, \phi_0(z_\tau)), \text{ as } N, T \to \infty, Th_{NT} \to \infty, h_{NT} \to 0,$$

which imply that, for any $z_\tau \in [0, 1]$, we have

$$\widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_\tau) \xrightarrow{P} \phi_0(z_\tau), \text{ as } N, T \to \infty, Th_{NT} \to \infty, h_{NT} \to 0.$$

This completes the proof. □

Proof of Theorem 2

Given the estimated curve $\widehat{\phi}_{\boldsymbol{\xi}_o NT}$, we have, for any $z_\tau \in [0, 1]$ and for $i = 1, \ldots, N$,

$$\frac{\partial}{\partial \phi} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i, \widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_\tau)) = \frac{\partial}{\partial \phi} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i, \phi_0(z_\tau)) + \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i, \bar{\phi}(z_\tau))(\widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_\tau) - \phi_0(z_\tau)), \tag{B-30}$$

where $\bar{\phi}(z_\tau)$ lies between $\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau)$ and $\phi_0(z_\tau)$. Then, taking the first order conditions of (B-23), from (B-30), for any $z_\tau \in [0,1]$, we have

$$
\begin{aligned}
0 \;=\;& \frac{1}{NTh_{NT}}\frac{\partial}{\partial\phi}\widetilde{\mathcal{L}}_{NT}(\boldsymbol{\xi}_0,\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau)) = \underbrace{\frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{K}\left(\frac{z_\tau-z_t}{h_{NT}}\right)\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_t))+}_{A_{NT}} \\
&+\underbrace{\frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{K}\left(\frac{z_\tau-z_t}{h_{NT}}\right)\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\bar{\phi}(z_t))\right](\phi_0(z_\tau)-\phi_0(z_t))+}_{D_{NT}} \\
&+\underbrace{\frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{K}\left(\frac{z_\tau-z_t}{h_{NT}}\right)\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_t))\right]}_{B_{NT}}(\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau)-\phi_0(z_\tau))+ \\
&+\underbrace{\frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{K}\left(\frac{z_\tau-z_t}{h_{NT}}\right)\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\bar{\phi}(z_t))\right]}_{C_{1NT}}(\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau)-\phi_0(z_\tau))+ \\
&-\underbrace{\frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{K}\left(\frac{z_0-z_t}{h_{NT}}\right)\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_t))\right]}_{C_{2NT}}(\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau)-\phi_0(z_\tau)). \qquad \text{(B-31)}
\end{aligned}
$$

By re-arranging (B-31), and defining $C_{NT} = C_{1NT} - C_{2NT}$, we obtain

$$
\sqrt{NTh_{NT}}(\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau)-\phi_0(z_\tau)) = -\frac{\sqrt{NTh_{NT}}(A_{NT}+D_{NT})}{(B_{NT}+C_{NT})}. \qquad \text{(B-32)}
$$

Let us consider each term on the right hand side of (B-32) separately.

Define $u_t = (z_\tau - z_t)/h_{NT}$, then, for $z_\tau \in (0,1)$, as $T\to\infty$ and $Th_{NT}\to\infty$, we have

$$
\frac{1}{Th_{NT}}\sum_{t=1}^{T} g(u_t)\mathrm{K}(u_t) \to \int_{-1}^{1} g(u)\mathrm{K}(u)\mathrm{d}u, \qquad \text{(B-33)}
$$

for $g(u) = u^p$, with $p = 0,\ldots,3$ and $g(u) = u^q\mathrm{K}(u)$ with $q = 0,\ldots,6$. All integrals in (B-33) are finite because of assumption K.

$\mathbf{A_{NT}}$. We have

$$\begin{aligned}
\mathrm{E}_0[A_{NT}] &= \frac{1}{NTh_{NT}}\mathrm{E}_0\left[\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_t))\right] = \\
&= \frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau + h_{NT}u_t))\mathrm{K}(u_t)\right] = \\
&= \frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{K}(u_t)\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right] + \\
&\quad + \frac{1}{NTh_{NT}}\sum_{t=1}^{T}h_{NT}u_t\phi_0'(z_\tau)\mathrm{K}(u_t)\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right].
\end{aligned}$$

If we use (B-33) and assumption K we have, as $T \to \infty$ and $Th_{NT} \to \infty$,

$$\mathrm{E}_0[A_{NT}] \to \frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right] = 0. \tag{B-34}$$

If N is fixed the expectation in (B-34) is zero by Lemma 1 in this paper. When $N \to \infty$, by assumption C.2, (B-34) is bounded for any N, and, again by Lemma 1 in this paper, we have, as $T \to \infty$, $Th_{NT} \to \infty$, and $N \to \infty$,

$$\mathrm{E}_0[A_{NT}] \to 0. \tag{B-35}$$

Then,

$$\begin{aligned}
\mathrm{E}_0\left[A_{NT}^2\right] &= \frac{1}{N^2T^2h_{NT}^2}\mathrm{E}_0\left[\left(\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_t))\right)^2\right] = \\
&= \frac{1}{N^2T^2h_{NT}^2}\sum_{t=1}^{T}\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau + h_{NT}u_t))\right)^2\mathrm{K}^2(u_t)\right] + \\
&\quad + \frac{1}{N^2T^2h_{NT}^2}\sum_{\substack{t,s=1\\ t\neq s}}^{T}\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau + h_{NT}u_t)\right)\mathrm{K}(u_t)\right. \\
&\quad \left.\left(\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{is}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau + h_{NT}u_s))\right)\mathrm{K}(u_s)\right] = \\
&= \mathrm{E}_0[A_{1NT}^2] + \mathrm{E}_0[A_{2NT}^2].
\end{aligned} \tag{B-36}$$

By using (B-33) and assumption K, we have, as $T \to \infty$ and $Th_{NT} \to \infty$,

$$\mathrm{E}_0[A_{2NT}^2] \to \frac{1}{N^2}\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau)\right)\left(\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{is}^{m}(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)\right] = 0, \tag{B-37}$$

by independence, since we are computing likelihoods in the true value of parameters, and Lemma 1 in this paper.

The other term in (B-36) is

$$
\begin{aligned}
\mathrm{E}_0\left[A_{1NT}^2\right] &= \frac{1}{N^2T^2h_{NT}^2}\sum_{t=1}^{T}\mathrm{K}^2(u_t)\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right]+\\
&\quad+\frac{1}{N^2T^2h_{NT}^2}\sum_{t=1}^{T}h_{NT}^2u_t^2\phi_0^{\prime 2}(z_\tau)\mathrm{K}^2(u_t)\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right].
\end{aligned}
$$

If we use (B-37), (B-33), and assumption K, we have, as $T\to\infty$ and $Th_{NT}\to\infty$,

$$
\begin{aligned}
\mathrm{E}_0\left[A_{NT}^2\right] &\to \frac{1}{N^2Th_{NT}}\kappa_1\,\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right]+\\
&+ \frac{h_{NT}}{N^2T}\phi_0^{\prime 2}(z_\tau)\kappa_2\,\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right]=\\
&= \frac{1}{NTh_{NT}}\bar{i}_{N\boldsymbol{\xi}_o}(z_\tau)\left[\kappa_1+h_{NT}^2\phi_0^{\prime 2}(z_\tau)\kappa_2\right]. \qquad \text{(B-38)}
\end{aligned}
$$

where we have defined

$$
\kappa_1=\int_{-1}^{1}\mathrm{K}^2(u)\mathrm{d}u,\quad \kappa_2=\int_{-1}^{1}u^2\mathrm{K}^2(u)\mathrm{d}u, \qquad \text{(B-39)}
$$

and

$$
\frac{1}{N^2}\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial}{\partial\phi}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right]=\frac{\bar{i}_{N\boldsymbol{\xi}_o}(z_\tau)}{N}.
$$

Therefore, since $\mathrm{E}_0[A_{NT}]\to 0$, as $T\to\infty$ and $Th_{NT}\to\infty$, we have

$$
\mathrm{Var}_0\left[A_{NT}\right]\to\frac{1}{NTh_{NT}}\bar{i}_{N\boldsymbol{\xi}_o}(z_\tau)\left[\kappa_1+h_{NT}^2\phi_0^{\prime 2}(z_\tau)\kappa_2\right]. \qquad \text{(B-40)}
$$

We can then apply the Weak Law of Large Numbers to A_{NT} which implies

$$
A_{NT}\xrightarrow{P}0,\ \text{ as }\ T\to\infty,\,Th_{NT}\to\infty. \qquad \text{(B-41)}
$$

If assumptions C.2 and D hold and we define $i_{\boldsymbol{\xi}_o}(z_\tau)=\lim_{N\to\infty}\bar{i}_{N\boldsymbol{\xi}_o}(z_\tau)$, this limit exists and is bounded. Thus when $T\to\infty$, $Th_{NT}\to\infty$, and $N\to\infty$

$$
\mathrm{Var}_0\left[A_{NT}\right]\to\frac{1}{NTh_{NT}}i_{\boldsymbol{\xi}_o}(z_\tau)\left[\kappa_1+h_{NT}^2\phi_0^{\prime 2}(z_\tau)\kappa_2\right]. \qquad \text{(B-42)}
$$

Analogously as in (B-41) we have

$$
A_{NT}\xrightarrow{P}0,\ \text{ as }\ N,T\to\infty,\,Th_{NT}\to\infty. \qquad \text{(B-43)}
$$

$\mathbf{B_{NT}}$. We have

$$\begin{aligned}
\mathrm{E}_0[B_{NT}] &= \frac{1}{NTh_{NT}}\mathrm{E}_0\left[\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_t))\right] = \\
&= \frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau + h_{NT}u_t))\mathrm{K}(u_t)\right] = \\
&= \frac{1}{NTh_{NT}}\sum_{t=1}^{T}\mathrm{K}(u_t)\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right] + \\
&\quad + \frac{1}{NTh_{NT}}\sum_{t=1}^{T}h_{NT}u_t\phi_0'(z_\tau)\mathrm{K}(u_t)\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial^3}{\partial\phi^3}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right].
\end{aligned}$$

If we use (B-33) and assumption K we have, as $T \to \infty$ and $Th_{NT} \to \infty$,

$$\mathrm{E}_0[B_{NT}] \to \frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right] = -\bar{\jmath}_{N\boldsymbol{\xi}_o}(z_\tau). \tag{B-44}$$

If assumption C.2 holds then $\lim_{N\to\infty}\bar{\jmath}_{N\boldsymbol{\xi}_o}(z_\tau) = j_{\boldsymbol{\xi}_o}(z_\tau)$ exists and is finite and, as $T \to \infty$, $Th_{NT} \to \infty$, and $N \to \infty$,

$$\mathrm{E}_0[B_{NT}] \to \frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right] = -j_{\boldsymbol{\xi}_o}(z_\tau).$$

Then let us compute the variance. We have

$$\begin{aligned}
\mathrm{E}_0[B_{NT}^2] &= \frac{1}{N^2T^2h_{NT}^2}\mathrm{E}_0\left[\left(\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_t))\right)^2\right] = \\
&= \frac{1}{N^2T^2h_{NT}^2}\sum_{t=1}^{T}\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau + h_{NT}u_t))\right)^2\mathrm{K}^2(u_t)\right] + \\
&\quad + \frac{1}{N^2T^2h_{NT}^2}\sum_{\substack{t,s=1\\ t\neq s}}^{T}\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau + h_{NT}u_t))\right)\mathrm{K}(u_t)\right. \\
&\quad \left.\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{is}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau + h_{NT}u_s))\right)\mathrm{K}(u_s)\right] = \\
&= \mathrm{E}_0[B_{1NT}^2] + \mathrm{E}_0[B_{2NT}^2].
\end{aligned} \tag{B-45}$$

By using (B-33) and assumption K, we can prove that, as $T \to \infty$ and $Th_{NT} \to \infty$,

$$\mathrm{E}_0[B_{2NT}^2] \to \bar{\jmath}_{N\boldsymbol{\xi}_o}^2(z_\tau). \tag{B-46}$$

The other term in (B-45) becomes

$$\begin{aligned}
\mathrm{E}_0[B_{1NT}^2] &= \frac{1}{N^2T^2h_{NT}^2}\sum_{t=1}^{T}\mathrm{K}^2(u_t)\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right]+\\
&+\frac{1}{N^2T^2h_{NT}^2}\sum_{t=1}^{T}h_{NT}^2u_t^2\phi_0^{'2}(z_\tau)\mathrm{K}^2(u_t)\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^3}{\partial\phi^3}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right]+\\
&+\frac{2}{N^2T^2h_{NT}^2}\sum_{t=1}^{T}h_{NT}u_t\phi_0^{'}(z_\tau)\mathrm{K}^2(u_t)\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)\right.\\
&\left.\left(\sum_{i=1}^{N}\frac{\partial^3}{\partial\phi^3}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)\right].
\end{aligned}$$

If we use (B-33) and assumption K we have, as $T\to\infty$ and $Th_{NT}\to\infty$,

$$\begin{aligned}
\mathrm{E}_0\left[B_{1NT}^2\right] &\to \frac{1}{N^2Th_{NT}}\kappa_1\,\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right]+\\
&+ \frac{h_{NT}}{N^2T}\phi_0^{'2}(z_\tau)\kappa_2\,\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^3}{\partial\phi^3}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right]=\\
&= \frac{1}{Th_{NT}}\kappa_1\bar{\mathcal{S}}_N(z_\tau)+\frac{h_{NT}}{T}\phi_0^{'2}(z_\tau)\kappa_2\bar{\mathcal{Q}}_N(z_\tau),
\end{aligned} \tag{B-47}$$

where κ_1 and κ_2 are defined in (B-39),

$$\kappa_3=\int_{-1}^{1}u^2\mathrm{K}(u)\mathrm{d}u, \tag{B-48}$$

and

$$\begin{aligned}
\bar{\mathcal{S}}_N(z_\tau) &= \frac{1}{N^2}\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right],\\
\bar{\mathcal{Q}}_N(z_\tau) &= \frac{1}{N^2}\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^3}{\partial\phi^3}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right].
\end{aligned} \tag{B-49}$$

Then, by combining (B-44), (B-45), (B-46), and (B-47), we have, as $T\to\infty$ and $Th_{NT}\to\infty$,

$$\mathrm{Var}_0[B_{NT}]\to\frac{1}{Th_{NT}}\kappa_1\bar{\mathcal{S}}_N(z_\tau)+\frac{h_{NT}}{T}\phi_0^{'2}(z_\tau)\kappa_2\bar{\mathcal{Q}}_N(z_\tau).$$

We can then apply the Weak Law of Large Numbers to B_{NT} which implies

$$B_{NT}\xrightarrow{P}-\bar{\jmath}_{N\boldsymbol{\xi}_o}(z_\tau)\ \text{ as }\ T\to\infty,\,Th_{NT}\to\infty. \tag{B-50}$$

Moreover, if assumption C.2 holds, then $j_{\boldsymbol{\xi}_o}(z_\tau) = \lim_{N\to\infty} \bar{j}_{N\boldsymbol{\xi}_o}(z_\tau)$, $\mathcal{S}(z_\tau) = \lim_{N\to\infty} \bar{\mathcal{S}}_N(z_\tau)$ and $\mathcal{Q}(z_\tau) = \lim_{N\to\infty} \bar{\mathcal{Q}}_N(z_\tau)$ exist and are finite. Therefore,

$$B_{NT} \xrightarrow{P} -j_{\boldsymbol{\xi}_o}(z_\tau) \text{ as } N, T \to \infty, \, Th_{NT} \to \infty. \tag{B-51}$$

$\mathbf{C_{NT}}$. We have

$$C_{NT} = \frac{1}{NTh_{NT}} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \frac{\partial^2}{\partial \phi^2} \left(\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i0}, \bar{\phi}(z_t)) - \boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i0}, \phi_0(z_t))\right).$$

By Theorem 1 we know that, when N is fixed,

$$\widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_\tau) \xrightarrow{P} \phi_0(z_\tau), \text{ as } T \to \infty, \, Th_{NT} \to \infty.$$

and since for any $z_\tau \in [0,1]$, we have $|\bar{\phi}(z_\tau) - \phi_0(z_\tau)| \leq |\widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_\tau) - \phi_0(z_\tau)|$, we also have

$$\bar{\phi}(z_\tau) \xrightarrow{P} \phi_0(z_\tau), \text{ as } T \to \infty, \, Th_{NT} \to \infty, \tag{B-52}$$

which implies

$$\begin{aligned}
|C_{NT}| &\leq \frac{1}{NTh_{NT}} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \left| \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i0}, \bar{\phi}(z_t)) - \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i0}, \phi_0(z_t)) \right| \leq \\
&\leq \frac{1}{NTh_{NT}} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \left| \frac{\partial^3}{\partial \phi^3} \boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i0}, \phi_0(z_t)) \right| \left| \bar{\phi}(z_\tau) - \phi_0(z_\tau) \right| \leq \\
&\leq \frac{1}{NTh_{NT}} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \left| \frac{\partial^3}{\partial \phi^3} \boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i0}, \phi_0(z_t)) \right| \left| \widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_\tau) - \phi_0(z_\tau) \right|.
\end{aligned} \tag{B-53}$$

Therefore, by using (B-52), we have

$$|C_{NT}| \xrightarrow{P} 0 \text{ as } T \to \infty, \, Th_{NT} \to \infty. \tag{B-54}$$

If assumption C.2 holds then all terms in (B-53) are bounded even when we let $N \to \infty$. Then, by using Theorem 1, we have

$$|C_{NT}| \xrightarrow{P} 0 \text{ as } N, T \to \infty, \, Th_{NT} \to \infty. \tag{B-55}$$

$\mathbf{D_{NT}}$. This is the bias term and it can be decomposed as

$$\begin{aligned}
D_{NT} &= \frac{1}{NTh_{NT}} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i0}, \bar{\phi}(z_t))(\phi_0(z_\tau) - \phi_0(z_t)) = \\
&= \frac{1}{NTh_{NT}} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i0}, \phi_0(z_t))(\phi_0(z_\tau) - \phi_0(z_t)) + \\
&\quad + \frac{1}{NTh_{NT}} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \frac{\partial^3}{\partial \phi^3} \boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i0}, \phi_0(z_t))(\bar{\phi}(z_t) - \phi_0(z_t))(\phi_0(z_\tau) - \phi_0(z_t)) = \\
&= D_{1NT} + D_{2NT}.
\end{aligned}$$

Since for any $z_\tau \in [0,1]$, we have $|\bar{\phi}(z_\tau) - \phi_0(z_\tau)| \leq |\widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_\tau) - \phi_0(z_\tau)|$, we have

$$|D_{2NT}| \xrightarrow{P} 0 \ \text{ as } \ T \to \infty, \, Th_{NT} \to \infty, \tag{B-56}$$

by Theorem 1 and (B-52). If assumption C.2 holds this is true even when $N \to \infty$.

Let us consider D_{1NT}, we have:

$$\begin{aligned}
\mathrm{E}_0[D_{1NT}] &= \frac{1}{NTh_{NT}} \mathrm{E}_0 \left[\sum_{t=1}^{T} \sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}} \right) \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_{i\,0}, \phi_0(z_t))(\phi_0(z_\tau) - \phi_0(z_t)) \right] = \\
&= \frac{1}{NTh_{NT}} \sum_{t=1}^{T} \mathrm{E}_0 \left[\sum_{i=1}^{N} \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_{i\,0}, \phi_0(z_\tau + h_{NT} u_t))(\phi_0(z_\tau) - \phi_0(z_\tau + h_{NT} u_t)) \mathrm{K}(u_t) \right] = \\
&= \frac{1}{NTh_{NT}} \sum_{t=1}^{T} \left(h_{NT} u_t \phi_0'(z_\tau) + \frac{h^2_{NT} u_t^2}{2} \phi_0^{''}(z_\tau) \right) \mathrm{K}(u_t) \left\{ \mathrm{E}_0 \left[\sum_{i=1}^{N} \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_{i\,0}, \phi_0(z_\tau)) \right] + \right. \\
&\quad \left. + \mathrm{E}_0 \left[\sum_{i=1}^{N} \frac{\partial^3}{\partial \phi^3} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_{i\,0}, \phi_0(z_\tau)) h_{NT} u_t \phi_0'(z_\tau)) \right] \right\} = \\
&= \frac{1}{NTh_{NT}} \sum_{t=1}^{T} h_{NT} u_t \phi_0'(z_\tau) \mathrm{K}(u_t) \mathrm{E}_0 \left[\sum_{i=1}^{N} \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_{i\,0}, \phi_0(z_\tau)) \right] + \\
&\quad + \frac{1}{NTh_{NT}} \sum_{t=1}^{T} \frac{h^2_{NT} u_t^2}{2} \phi_0^{''}(z_\tau) \mathrm{K}(u_t) \mathrm{E}_0 \left[\sum_{i=1}^{N} \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_{i\,0}, \phi_0(z_\tau)) \right] + \\
&\quad + \frac{1}{NTh_{NT}} \sum_{t=1}^{T} h^2_{NT} u_t^2 \phi_0^{'2}(z_\tau) \mathrm{K}(u_t) \mathrm{E}_0 \left[\sum_{i=1}^{N} \frac{\partial^3}{\partial \phi^3} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_{i\,0}, \phi_0(z_\tau)) \right] + \\
&\quad + \frac{1}{NTh_{NT}} \sum_{t=1}^{T} \frac{h^3_{NT} u_t^3}{2} \phi_0^{''}(z_\tau) \phi_0'(z_\tau) \mathrm{K}(u_t) \mathrm{E}_0 \left[\sum_{i=1}^{N} \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_{i\,0}, \phi_0(z_\tau)) \right].
\end{aligned}$$

If we use (B-33) and assumption K we have, as $T \to \infty$ and $Th_{NT} \to \infty$,

$$\begin{aligned}
\mathrm{E}_0[D_{1NT}] &\to h^2_{NT} \kappa_3 \frac{\phi_0^{''}(z_\tau)}{2} \frac{1}{N} \mathrm{E}_0 \left[\sum_{i=1}^{N} \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_{i\,0}, \phi_0(z_\tau)) \right] + \\
&+ h^2_{NT} \kappa_3 \phi_0^{'2}(z_\tau) \frac{\partial}{\partial \phi} \frac{1}{N} \mathrm{E}_0 \left[\sum_{i=1}^{N} \frac{\partial^2}{\partial \phi^2} \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_{i\,0}, \phi_0(z_\tau)) \right] = \\
&= h^2_{NT} \kappa_3 \left[-\frac{\phi_0^{''}(z_\tau)}{2} \bar{\jmath}_{N\boldsymbol{\xi}_o}(z_\tau) - \phi_0^{'2}(z_\tau) \frac{\partial}{\partial \phi} \bar{\jmath}_{N\boldsymbol{\xi}_o}(z_\tau) \right] = \\
&= h^2_{NT} \kappa_3 \bar{\mathcal{B}}_N(z_\tau),
\end{aligned} \tag{B-57}$$

where $\bar{\jmath}_{N\boldsymbol{\xi}_o}(z_\tau)$ is defined in (B-44), κ_3 is defined in (B-48) and

$$\bar{\mathcal{B}}_N(z_\tau) = -\bar{\jmath}_{N\boldsymbol{\xi}_o}(z_\tau) \frac{\phi_0^{''}(z_\tau)}{2} - \frac{\partial}{\partial \phi} \bar{\jmath}_{N\boldsymbol{\xi}_o}(z_\tau) \phi_0^{'2}(z_\tau).$$

Now, let us compute the variance. We have

$$
\begin{aligned}
\mathrm{E}_0\left[D_{1NT}^2\right] &= \frac{1}{N^2T^2h_{NT}^2}\mathrm{E}_0\left[\left(\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_t))(\phi_0(z_t)-\phi_0(z_\tau))\right)^2\right] = \\
&= \frac{1}{N^2T^2h_{NT}^2}\sum_{t=1}^{T}(\phi_0(z_\tau)-\phi_0(z_\tau+h_{NT}u_t))^2\,\mathrm{K}^2(u_t) \\
&\quad \mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau+h_{NT}u_t))\right)^2\right] + \\
&\quad +\frac{1}{N^2T^2h_{NT}^2}\sum_{\substack{t,s=1\\ t\neq s}}^{T}(\phi_0(z_\tau)-\phi_0(z_\tau+h_{NT}u_t))\,(\phi_0(z_\tau)-\phi_0(z_\tau+h_{NT}u_s))\,\mathrm{K}(u_t)\mathrm{K}(u_s) \\
&\quad \mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau+h_{NT}u_t))\right)\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau+h_{NT}u_s))\right)\right] = \\
&= \mathrm{E}_0[D_{1,1NT}^2] + \mathrm{E}_0[D_{1,2NT}^2]. \qquad \text{(B-58)}
\end{aligned}
$$

By using (B-33) and assumption K, we can prove that, as $T\to\infty$ and $Th_{NT}\to\infty$,

$$
\mathrm{E}_0[D_{1,2NT}^2] \to O(h_{NT}^3). \qquad \text{(B-59)}
$$

The other term in (B-58) becomes

$$
\mathrm{E}_0[D_{1,1NT}^2] = \frac{1}{N^2T^2h_{NT}^2}\sum_{t=1}^{T}h_{NT}^2u_t^2\phi_0'^2(z_\tau)\mathrm{K}^2(u_t)\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right]+O(h_{NT}^3).
$$

If we use (B-33) and assumption K we have, as $T\to\infty$ and $Th_{NT}\to\infty$,

$$
\begin{aligned}
\mathrm{E}_0[D_{1,1NT}^2] &\to \frac{h_{NT}}{N^2T}\phi_0'^2(z_\tau)\kappa_2\,\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi^2}\boldsymbol{\ell}_{it}^m(\boldsymbol{\xi}_{i\,0},\phi_0(z_\tau))\right)^2\right] + O(h_{NT}^3) = \\
&= \frac{h_{NT}}{T}\phi_0'^2(z_\tau)\kappa_2\bar{\mathcal{S}}_N(z_\tau) + O(h_{NT}^3), \qquad \text{(B-60)}
\end{aligned}
$$

where κ_2 is defined in (B-39) and $\bar{\mathcal{S}}_N(z_\tau)$ is defined in (B-49).

Therefore, by combining (B-57), (B-58), (B-59), and (B-60), and keeping only terms up to $O(h_{NT}^2)$ we have, as $T\to\infty$ and $Th_{NT}\to\infty$,

$$
\mathrm{Var}_0\left[D_{1NT}\right] \to \frac{h_{NT}}{T}\phi_0'^2(z_\tau)\kappa_2\bar{\mathcal{S}}_N(z_\tau). \qquad \text{(B-61)}
$$

Since by (B-56), $D_{NT} = D_{1NT} + o_P(1)$, as $T\to\infty$ and $Th_{NT}\to\infty$, we can then apply the Weak Law of Large Numbers to D_{NT} which implies

$$
D_{NT} \xrightarrow{P} h_{NT}^2\kappa_3\bar{\mathcal{B}}_N(z_\tau), \;\text{ as }\; T\to\infty, \quad Th_{NT}\to\infty. \qquad \text{(B-62)}
$$

Moreover, if assumption C.2 holds, then $\mathcal{B}(z_\tau) = \lim_{N\to\infty} \bar{\mathcal{B}}_N(z_\tau)$ and $\mathcal{S}(z_\tau) = \lim_{N\to\infty} \bar{\mathcal{S}}_N(z_\tau)$ exist and are finite. Thus, as $T \to \infty$, $Th_{NT} \to \infty$, and $N \to \infty$,

$$\mathrm{Var}_0\left[D_{1NT}\right] \to \frac{h_{NT}}{T}\phi_0'^2(z_\tau)\kappa_2\mathcal{S}(z_\tau). \tag{B-63}$$

and

$$D_{NT} \xrightarrow{P} h_{NT}^2\kappa_3\mathcal{B}(z_\tau), \text{ as } N, T \to \infty, \quad Th_{NT} \to \infty. \tag{B-64}$$

We now consider the limiting distribution of (B-32) up to terms of order $O(h_{NT}^2)$. First consider the case $i)$ in Theorem 2, i.e. when N is fixed and T is large. Then, from (B-32) and using (B-41) and (B-62) we have

$$\left(\widehat{\phi}_{\boldsymbol{\xi}_o T}(z_\tau) - \phi_0(z_\tau) - h_{NT}^2\bar{\mathcal{B}}_N(z_\tau)\kappa_3\right) \xrightarrow{P} 0 \text{ as } T \to \infty, Th_{NT} \to \infty.$$

By applying the Central Limit Theorem by Wooldridge and White (1988) and Slutsky's Theorem, we have convergence in distribution, as $T \to \infty$ and $Th_{NT} \to \infty$:

$$\sqrt{NTh_{NT}}\left(\widehat{\phi}_{\boldsymbol{\xi}_o T}(z_\tau) - \phi_0(z_\tau) - h_{NT}^2\bar{\mathcal{B}}_N(z_\tau)\kappa_3\right) \xrightarrow{d} \mathcal{N}\left(0, \widetilde{V}_{\boldsymbol{\xi}_o}(z_\tau) + W_{\boldsymbol{\xi}_o}(z_\tau) + U_{\boldsymbol{\xi}_o}(z_\tau)\right).$$

Using (B-40), (B-50), (B-54), and (B-61), the asymptotic variance is

$$\begin{aligned}
\widetilde{V}_{\boldsymbol{\xi}_o}(z_\tau) &= NTh_{NT}\frac{\mathrm{Var}_0\left[A_{NT}\right]}{\mathrm{E}_0\left[B_{NT}\right]^2} = \frac{\bar{i}_{N\,\boldsymbol{\xi}_o}(z_\tau)}{\bar{j}_{N\,\boldsymbol{\xi}_o}^2(z_\tau)}\left[\kappa_1 + h_{NT}^2\phi_0'^2(z_\tau)\kappa_2\right], \qquad \text{(B-65)} \\
W_{\boldsymbol{\xi}_o}(z_\tau) &= NTh_{NT}\frac{\mathrm{Var}_0\left[D_{NT}\right]}{\mathrm{E}_0\left[B_{NT}\right]^2} = Nh_{NT}^2\frac{\phi_0'^2(z_\tau)\bar{\mathcal{S}}_N(z_\tau)}{\bar{j}_{N\,\boldsymbol{\xi}_o}^2(z_\tau)}, \\
U_{\boldsymbol{\xi}_o}(z_\tau) &= NTh_{NT}\frac{2\,\mathrm{Cov}_0\left[A_{NT}, D_{NT}\right]}{\mathrm{E}_0\left[B_{NT}\right]^2}.
\end{aligned}$$

If we also let $h_{NT} \to 0$, and by the Cauchy–Schwarz inequality, we have

$$\widetilde{V}_{\boldsymbol{\xi}_o}(z_\tau) \to V_{\boldsymbol{\xi}_o}(z_\tau), \qquad W_{\boldsymbol{\xi}_o}(z_\tau) \to 0, \qquad U_{\boldsymbol{\xi}_o}(z_\tau) \leq \sqrt{2V_{\boldsymbol{\xi}_o}(z_\tau)W_{\boldsymbol{\xi}_o}(z_\tau)} \to 0,$$

where

$$V_{\boldsymbol{\xi}_o}(z_\tau) = \frac{\bar{i}_{N\,\boldsymbol{\xi}_o}(z_\tau)}{\bar{j}_{N\,\boldsymbol{\xi}_o}^2(z_\tau)}\kappa_1.$$

Moreover, also the bias term becomes negligible and we have

$$\sqrt{NTh_{NT}}\left(\widehat{\phi}_{\boldsymbol{\xi}_o T}(z_\tau) - \phi_0(z_\tau)\right) \xrightarrow{d} \mathcal{N}\left(0, V_{\boldsymbol{\xi}_o}(z_\tau)\right), \text{ as } T \to \infty, Th_{NT} \to \infty.$$

Now let us consider case $ii)$ in Theorem 2, i.e. when both N and T are large. By assuming that assumptions C.4 and D hold and using (B-43) and (B-64), we have from (B-32)

$$\left(\widehat{\phi}_{\boldsymbol{\xi}_o T}(z_\tau) - \phi_0(z_\tau) - h_{NT}^2\mathcal{B}(z_\tau)\kappa_3\right) \xrightarrow{P} 0 \text{ as } N, T \to \infty, NTh_{NT} \to \infty.$$

Then, by applying the Central Limit Theorem by Wooldridge and White (1988) and Slutsky's Theorem, we have convergence in distribution, as $N, T \to \infty$ and $NTh_{NT} \to \infty$:

$$\sqrt{NTh_{NT}}\left(\widehat{\phi}_{\boldsymbol{\xi}_o T}(z_\tau) - \phi_0(z_\tau) - h_{NT}^2\mathcal{B}(z_\tau)\kappa_3\right) \xrightarrow{d} \mathcal{N}\left(0, \widetilde{V}_{\boldsymbol{\xi}_o}(z_\tau) + W_{\boldsymbol{\xi}_o}(z_\tau) + U_{\boldsymbol{\xi}_o}(z_\tau)\right).$$

Using (B-42), (B-51), (B-55), and (B-63) we have

$$\begin{aligned}
\widetilde{V}_{\boldsymbol{\xi}_o}(z_\tau) &= \frac{i_{\boldsymbol{\xi}_o}(z_\tau)}{j^2_{\boldsymbol{\xi}_o}(z_\tau)}\left[\kappa_1 + h^2_{NT}\phi_0'^2(z_\tau)\kappa_2\right], \qquad \text{(B-66)}\\
W_{\boldsymbol{\xi}_o}(z_\tau) &= Nh^2_{NT}\frac{\phi_0'^2(z_\tau)\mathcal{S}(z_\tau)}{j^2_{\boldsymbol{\xi}_o}(z_\tau)},\\
U_{\boldsymbol{\xi}_o}(z_\tau) &= \lim_{N\to\infty} NTh_{NT}\frac{2\,\mathrm{Cov}_0\left[A_{NT}, D_{NT}\right]}{\mathrm{E}_0\left[B_{NT}\right]^2}.
\end{aligned}$$

If we also let $Nh^2_{NT} \to 0$ which implies $h_{NT} \to 0$, and by the Cauchy–Schwarz inequality, we have

$$\widetilde{V}_{\boldsymbol{\xi}_o}(z_\tau) \to V_{\boldsymbol{\xi}_o}(z_\tau), \qquad W_{\boldsymbol{\xi}_o}(z_\tau) \to 0, \qquad U_{\boldsymbol{\xi}_o}(z_\tau) \le \sqrt{2V_{\boldsymbol{\xi}_o}(z_\tau)W_{\boldsymbol{\xi}_o}(z_\tau)} \to 0,$$

where

$$V_{\boldsymbol{\xi}_o}(z_\tau) = \frac{i_{\boldsymbol{\xi}_o}(z_\tau)}{j^2_{\boldsymbol{\xi}_o}(z_\tau)}\kappa_1.$$

Moreover, also the bias term becomes negligible and we have

$$\sqrt{NTh_{NT}}\left(\widehat{\phi}_{\boldsymbol{\xi}_o T}(z_\tau) - \phi_0(z_\tau)\right) \xrightarrow{d} \mathcal{N}\left(0, V_{\boldsymbol{\xi}_o}(z_\tau)\right), \text{ as } N,T\to\infty,\ NTh_{NT}\to\infty.$$

This completes the proof. □

Proof of Theorem 3

a) From the first order conditions of (B-23), for any $z_\tau \in [0,1]$ and any $\boldsymbol{\xi} \in \boldsymbol{\Xi}$, we have

$$\frac{\partial\widetilde{\mathcal{L}}_{NT}}{\partial\phi}(\boldsymbol{\xi}_0, \widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_\tau)) = 0.$$

Then compute the derivative with respect to $\boldsymbol{\xi}$, i.e., for any $z_\tau \in [0,1]$,

$$\begin{aligned}
\mathbf{0} = \frac{\partial}{\partial\boldsymbol{\xi}}\left(\frac{\partial\widetilde{\mathcal{L}}_{NT}}{\partial\phi}(\boldsymbol{\xi}_0, \widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_\tau))\right) &= \sum_{t=1}^T\sum_{i=1}^N \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2\boldsymbol{\ell}^m_{it}}{\partial\boldsymbol{\xi}\partial\phi}(\boldsymbol{\xi}_0, \widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_t)) + \\
&\quad + \sum_{t=1}^T\sum_{i=1}^N \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2\boldsymbol{\ell}^m_{it}}{\partial\phi^2}(\boldsymbol{\xi}_0, \widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_t))\,\widehat{\phi}'_{\boldsymbol{\xi}_o NT}(z_t).
\end{aligned}$$

Solving for $\widehat{\phi}'_{\boldsymbol{\xi}_o NT}$ we have, for any $z_\tau \in [0,1]$,

$$\widehat{\phi}'_{\boldsymbol{\xi}_o NT}(z_\tau) = -\frac{\frac{1}{NTh_{NT}}\sum_{t=1}^T\sum_{i=1}^N \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2\boldsymbol{\ell}^m_{it}}{\partial\boldsymbol{\xi}\partial\phi}(\boldsymbol{\xi}_0, \widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_t))}{\frac{1}{NTh_{NT}}\sum_{t=1}^T\sum_{i=1}^N \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2\boldsymbol{\ell}^m_{it}}{\partial\phi^2}(\boldsymbol{\xi}_0, \widehat{\phi}_{\boldsymbol{\xi}_o NT}(z_t))} = -\frac{\boldsymbol{\alpha}_{NT}(z_\tau)}{\beta_{NT}(z_\tau)}. \qquad \text{(B-67)}$$

Moreover, using a Taylor expansion in a neighborhood of ϕ_0, and for T sufficiently large,

$$\begin{aligned}
&\left|\left|\boldsymbol{\alpha}_{NT}(z_\tau) - \frac{1}{NTh_{NT}}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2\boldsymbol{\ell}_{it}^m}{\partial\boldsymbol{\xi}\partial\phi}(\boldsymbol{\xi}_0, \phi_0(z_t))\right|\right|_2 = \\
&= \left|\left|\frac{1}{NTh_{NT}}\frac{\partial}{\partial\phi}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2\boldsymbol{\ell}_{it}^m}{\partial\boldsymbol{\xi}\partial\phi}(\boldsymbol{\xi}_0, \bar{\phi}(z_t))\,(\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_t) - \phi_0(z_t))\right|\right|_2 \leq \qquad \text{(B-68)}\\
&\leq \left|\left|\frac{1}{NTh_{NT}}\frac{\partial}{\partial\phi}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2\boldsymbol{\ell}_{it}^m}{\partial\boldsymbol{\xi}\partial\phi}(\boldsymbol{\xi}_0, \bar{\phi}(z_t))\right|\right|_2 \sup_{z_t\in[0,1]}\left|\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_t) - \phi_0(z_t)\right| = o_P(1),
\end{aligned}$$

where $\bar{\phi}$ is between $\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}$ and ϕ_0. The previous result is a consequence of Theorem 1 on the consistency of $\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}$ and assumption S. Similarly we can prove, for any $z_\tau \in [0,1]$, and for T sufficiently large,

$$\left|\beta_{NT}(z_\tau) - \frac{1}{NTh_{NT}}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2\boldsymbol{\ell}_{it}^m}{\partial\phi^2}(\boldsymbol{\xi}_0, \phi_0(z_t))\right| = o_P(1). \qquad \text{(B-69)}$$

Define,

$$\begin{aligned}
\boldsymbol{\alpha}_{NT}^*(z_\tau) &= \frac{1}{NTh_{NT}}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2\boldsymbol{\ell}_{it}^m}{\partial\boldsymbol{\xi}\partial\phi}(\boldsymbol{\xi}_0, \phi_0(z_t)),\\
\beta_{NT}^*(z_\tau) &= \frac{1}{NTh_{NT}}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right)\frac{\partial^2\boldsymbol{\ell}_{it}^m}{\partial\phi^2}(\boldsymbol{\xi}_0, \phi_0(z_t)).
\end{aligned}$$

Using calculations similar to those in the proof of Theorem 2 and by applying the Weak Law of Large Numbers, we have, for any $z_\tau \in [0,1]$, as $T \to \infty$, $Th_{NT} \to \infty$, and $h_{NT} \to 0$,

$$\begin{aligned}
\boldsymbol{\alpha}_{NT}^*(z_\tau) &\xrightarrow{P} \frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial^2\boldsymbol{\ell}_{it}^m}{\partial\boldsymbol{\xi}\partial\phi}(\boldsymbol{\xi}_0, \phi_0(z_\tau))\right] = \bar{\boldsymbol{d}}_{N\,\boldsymbol{\xi}_o}(z_\tau),\\
\beta_{NT}^*(z_\tau) &\xrightarrow{P} \frac{1}{N}\mathrm{E}_0\left[\sum_{i=1}^{N}\frac{\partial^2\boldsymbol{\ell}_{it}^m}{\partial\phi^2}(\boldsymbol{\xi}_0, \phi_0(z_\tau))\right] = -\bar{j}_{N\,\boldsymbol{\xi}_o}(z_\tau).
\end{aligned} \qquad \text{(B-70)}$$

By combining, (B-68) and (B-69) with (B-70), we have, as $T \to \infty$, $Th_{NT} \to \infty$, and $h_{NT} \to 0$,

$$\begin{aligned}
\left|\left|\boldsymbol{\alpha}_{NT}(z_\tau) - \bar{\boldsymbol{d}}_{N\,\boldsymbol{\xi}_o}(z_\tau)\right|\right|_2 &\leq ||\boldsymbol{\alpha}_{NT}(z_\tau) - \boldsymbol{\alpha}_{NT}^*(z_\tau)||_2 + \left|\left|\boldsymbol{\alpha}_{NT}^*(z_\tau) - \bar{\boldsymbol{d}}_{N\,\boldsymbol{\xi}_o}(z_\tau)\right|\right|_2 = o_P(1),\\
|\beta_{NT}(z_\tau) - (-\bar{j}_{N\,\boldsymbol{\xi}_o}(z_\tau))| &\leq |\beta_{NT}(z_\tau) - \beta_{NT}^*(z_\tau)| + |\beta_{NT}^*(z_\tau) - (-\bar{j}_{N\,\boldsymbol{\xi}_o}(z_\tau))| = o_P(1),
\end{aligned}$$

which, substituted in (B-67), implies

$$\widehat{\phi}'_{\boldsymbol{\xi}_o\,NT}(z_\tau) \xrightarrow{P} \frac{\bar{\boldsymbol{d}}_{N\,\boldsymbol{\xi}_o}(z_\tau)}{\bar{j}_{N\,\boldsymbol{\xi}_o}(z_\tau)}, \text{ as } T \to \infty,\, Th_{NT} \to \infty,\, h_{NT} \to 0. \qquad \text{(B-71)}$$

b) If assumption C.2 holds then we know that $\lim_{N\to\infty}\bar{j}_{N\,\boldsymbol{\xi}_o}(z_\tau) = j_{\boldsymbol{\xi}_o}(z_\tau)$ exists and is finite.

Concerning the first term in (B-70), since each marginal depends only on its parameters $\boldsymbol{\xi}_i$, we have

$$\begin{aligned}
\bar{\boldsymbol{d}}_{N\,\boldsymbol{\xi}_o}(z_\tau) &= \frac{1}{N}\left\{ \mathrm{E}_0 \begin{bmatrix} \frac{\partial^2 \boldsymbol{\ell}^m_{1\,t}}{\partial \boldsymbol{\xi}_1 \partial \phi}(\boldsymbol{\xi}_{1\,0}, \phi_0(z_\tau)) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} + \ldots + \mathrm{E}_0 \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \frac{\partial^2 \boldsymbol{\ell}^m_{N\,t}}{\partial \boldsymbol{\xi}_N \partial \phi}(\boldsymbol{\xi}_{N\,0}, \phi_0(z_\tau)) \end{bmatrix} \right\} = \\
&= \frac{1}{N}\mathrm{E}_0 \begin{bmatrix} \frac{\partial^2 \boldsymbol{\ell}^m_{1\,t}}{\partial \boldsymbol{\xi}_1 \partial \phi}(\boldsymbol{\xi}_{1\,0}, \phi_0(z_\tau)) \\ \vdots \\ \vdots \\ \frac{\partial^2 \boldsymbol{\ell}^m_{N\,t}}{\partial \boldsymbol{\xi}_N \partial \phi}(\boldsymbol{\xi}_{N\,0}, \phi_0(z_\tau)) \end{bmatrix}. \qquad \text{(B-72)}
\end{aligned}$$

Therefore, $\lim_{N\to\infty} \bar{\boldsymbol{d}}_{N\,\boldsymbol{\xi}_o}(z_\tau) = \mathbf{0}$, for any $z_\tau \in [0,1]$. This completes the proof. $\square$

Proof of Corollary 1

The proof follows the same steps as in Lemma 4 and 5 in Severini and Wong (1992). We define $\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)$ such that it solves (B-23) when computed in generic values of the parameters $\boldsymbol{\xi} \in \Xi$, i.e., for any $z_\tau \in [0,1]$,

$$\frac{\partial}{\partial \phi}\left\{ \frac{1}{NTh_{NT}} \sum_{t=1}^{T}\sum_{i=1}^{N} \mathrm{K}\left(\frac{z_\tau - z_t}{h_{NT}}\right) \boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i, \widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_t)) \right\} = 0. \qquad \text{(B-73)}$$

According to assumption N, the limit in probability of $\widehat{\phi}_{\boldsymbol{\xi}\,NT}$ exists and we denote it as $\widetilde{\phi}_{\boldsymbol{\xi}}$. In order to prove the Corollary, we have to show that assumption N holds and that $\widetilde{\phi}_{\boldsymbol{\xi}}$ is a least favorable curve, i.e. its derivative is equal to the least favorable direction, which for $\boldsymbol{\xi} = \boldsymbol{\xi}_0$ is defined in (18).

First, notice that, if $\boldsymbol{\xi} = \boldsymbol{\xi}_0$, the result is a direct consequence of Theorems 1 and 3. Indeed, from Theorem 1.a, we know that, for any $z_\tau \in [0,1]$,

$$\widehat{\phi}_{\boldsymbol{\xi}_o\,NT}(z_\tau) \xrightarrow{P} \phi_0, \text{ as } T \to \infty, Th_{NT} \to \infty, h_{NT} \to 0.$$

Therefore, $\widetilde{\phi}_{\boldsymbol{\xi}_o} = \phi_0$. Moreover, from Theorem 3.a, we have that, for any $z_\tau \in [0,1]$,

$$\widehat{\phi}'_{\boldsymbol{\xi}_o\,NT}(z_\tau) \xrightarrow{P} \phi'_0(z_\tau), \text{ as } T \to \infty, Th_{NT} \to \infty, h_{NT} \to 0,$$

which implies $\widetilde{\phi}'_{\boldsymbol{\xi}_o} = \phi'_0$. Thus, $\widetilde{\phi}_{\boldsymbol{\xi}_o} = \phi_0$ is a least favorable curve.

Now let us move to the case in which we consider a generic value of the parameters $\boldsymbol{\xi} \in \Xi$. Lemma 4 in this paper proves that, assumption N is always satisfied, i.e. there exists a curve $\widetilde{\phi}_{\boldsymbol{\xi}}$ such that, as $T \to \infty$, $Th_{NT} \to \infty$, $h_{NT} \to 0$, for any $z_\tau \in [0,1]$ and any $\boldsymbol{\xi} \in \Xi$,

$$\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) \xrightarrow{P} \widetilde{\phi}_{\boldsymbol{\xi}}(z_\tau), \qquad \widehat{\phi}'_{\boldsymbol{\xi}\,NT}(z_\tau) \xrightarrow{P} \widetilde{\phi}'_{\boldsymbol{\xi}}(z_\tau).$$

In particular, (B-18) and (B-21) in Lemma 4 show that, for any $z_\tau \in [0,1]$ and any $\boldsymbol{\xi} \in \Xi$, as

$T \to \infty, Th_{NT} \to \infty, h_{NT} \to 0,$

$$\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau) \xrightarrow{P} \phi_{\boldsymbol{\xi}}(z_\tau), \qquad \widehat{\phi}'_{\boldsymbol{\xi}\,NT}(z_\tau) \xrightarrow{P} \phi'_{\boldsymbol{\xi}}(z_\tau), \tag{B-74}$$

where $\phi_{\boldsymbol{\xi}}(z_\tau)$ is such that it solves (B-25), i.e., for any $z_\tau \in [0,1]$ and any fixed $\boldsymbol{\xi} \in \Xi$,

$$\frac{\partial}{\partial \phi}\left\{\frac{1}{N}\sum_{i=1}^{N} \mathrm{E}_0\left[\boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_i, \phi_{\boldsymbol{\xi}}(z_\tau))\right]\right\} = 0. \tag{B-75}$$

Therefore, given (B-74) and Lemma 4, assumption N is always satisfied with $\widetilde{\phi}_{\boldsymbol{\xi}} = \phi_{\boldsymbol{\xi}}$.

Finally, by using the same arguments as in Theorem 3.a, and a Taylor series approximation of (B-75) in a neighborhood of $\phi_{\boldsymbol{\xi}}$, we can prove that

$$\phi'_{\boldsymbol{\xi}} = \frac{\bar{\boldsymbol{d}}_{N\,\boldsymbol{\xi}}(z_\tau)}{\bar{\jmath}_{N\,\boldsymbol{\xi}}(z_\tau)}, \text{ as } T \to \infty, Th_{NT} \to \infty, h_{NT} \to 0, \tag{B-76}$$

where $\bar{\jmath}_{N\,\boldsymbol{\xi}}(z_\tau)$ and $\bar{\boldsymbol{d}}_{N\,\boldsymbol{\xi}}(z_\tau)$ are analogous to the ones defined in Theorem 3.a. Therefore, by comparing (B-76) with (18) in the paper, we recognize $\phi'_{\boldsymbol{\xi}}$ as a least favorable direction, which implies that $\phi_{\boldsymbol{\xi}}$ is a least favorable curve and $\widehat{\phi}_{\boldsymbol{\xi}\,NT}$ is its estimator. To conclude also notice that, when considering $\boldsymbol{\xi} = \boldsymbol{\xi}_0$, we have $\phi_0 = \phi_{\boldsymbol{\xi}_o}$ and $\phi'_0 = \phi'_{\boldsymbol{\xi}_o}$ where the latter is defined in equation (18). This completes the proof. □

Proof of Theorem 4

a) Given the estimator $\widehat{\phi}_{\boldsymbol{\xi}\,NT}$ in (12), we define, for any $i = 1, \ldots, N$,

$$\mathcal{L}^m_{i\,T}(\boldsymbol{\xi}_i, \widehat{\phi}_{\boldsymbol{\xi}\,NT}) = \sum_{t=1}^{T} \boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_i, \widehat{\phi}_{\boldsymbol{\xi}\,NT}).$$

We first prove consistency of $\widehat{\boldsymbol{\xi}}_{i\,T}$, such that (see (19)),

$$\widehat{\boldsymbol{\xi}}_{i\,T} = \arg\max_{\boldsymbol{\xi}_i \in \Xi_i} \mathcal{L}^m_{i\,T}(\boldsymbol{\xi}_i, \widehat{\phi}_{\boldsymbol{\xi}\,NT}). \tag{B-77}$$

We also define the function

$$\gamma_i(\boldsymbol{\xi}_i) = \mathrm{E}_0\left[\boldsymbol{\ell}^m_{i\,t}(\boldsymbol{\xi}_i, \phi_{\boldsymbol{\xi}})\right],$$

where $\phi_{\boldsymbol{\xi}}$ is the least favorable curve computed in a generic value of the parameters $\boldsymbol{\xi}$. From Lemma 1 and assumption H the true value of the parameters is such that

$$\boldsymbol{\xi}_{i\,0} = \arg\max_{\boldsymbol{\xi}_i \in \Xi_i} \gamma_i(\boldsymbol{\xi}_i). \tag{B-78}$$

Then, following the same argument as in the proof of Lemma 4 and using (B-13), we can use the Weak Law of Large Numbers by McLeish (1975) and Lemma 2.1 of White and Domowitz (1984), which imply that, for any $\boldsymbol{\xi}_i \in \Xi_i$,

$$\frac{1}{T}\mathcal{L}^m_{i\,T}(\boldsymbol{\xi}_i, \phi_{\boldsymbol{\xi}}) \xrightarrow{P} \gamma_i(\boldsymbol{\xi}_i), \text{ as } T \to \infty. \tag{B-79}$$

Moreover,

$$\frac{1}{T}\left|\mathcal{L}^m_{iT}(\boldsymbol{\xi}_i,\widehat{\phi}_{\boldsymbol{\xi}\,NT})-\mathcal{L}^m_{iT}(\boldsymbol{\xi}_i,\phi_{\boldsymbol{\xi}})\right|\le\frac{1}{T}\sum_{t=1}^T\left|\boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i,\widehat{\phi}_{\boldsymbol{\xi}\,NT})-\boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i,\phi_{\boldsymbol{\xi}})\right|=$$

$$=\frac{1}{T}\sum_{t=1}^T\left|\frac{\partial}{\partial\phi}\boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i,\phi_{\boldsymbol{\xi}})\right|\sup_{z_\tau\in[0,1]}\left|\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)-\phi_{\boldsymbol{\xi}}(z_\tau)\right|\le$$

$$\le\frac{1}{T}\sum_{t=1}^T\sup_{\boldsymbol{\xi}_i\in\boldsymbol{\Xi}_i}\left|\frac{\partial}{\partial\phi}\boldsymbol{\ell}^m_{it}(\boldsymbol{\xi}_i,\phi_{\boldsymbol{\xi}})\right|\sup_{\boldsymbol{\xi}_i\in\boldsymbol{\Xi}_i}\sup_{z_\tau\in[0,1]}\left|\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)-\phi_{\boldsymbol{\xi}}(z_\tau)\right|.$$

From Theorem 1 we have (notice that if assumption C.2 holds the following is true also when $N\to\infty$)

$$\sup_{z_\tau\in[0,1]}\left|\widehat{\phi}_{\boldsymbol{\xi}\,NT}(z_\tau)-\phi_{\boldsymbol{\xi}}(z_\tau)\right|=o_P(1),\ \text{ as }\ T\to\infty,$$

therefore, for any $\boldsymbol{\xi}_i\in\boldsymbol{\Xi}_i$,

$$\frac{1}{T}\left|\mathcal{L}^m_{iT}(\boldsymbol{\xi}_i,\widehat{\phi}_{\boldsymbol{\xi}\,NT})-\mathcal{L}^m_{iT}(\boldsymbol{\xi}_i,\phi_{\boldsymbol{\xi}})\right|\xrightarrow{P}0,\ \text{ as }\ T\to\infty. \tag{B-80}$$

By combining (B-79) and (B-80), we have, for any $\boldsymbol{\xi}_i\in\boldsymbol{\Xi}_i$,

$$\frac{1}{T}\mathcal{L}^m_{iT}(\boldsymbol{\xi}_i,\widehat{\phi}_{\boldsymbol{\xi}\,NT})\xrightarrow{P}\gamma_i(\boldsymbol{\xi}_i),\ \text{ as }\ T\to\infty. \tag{B-81}$$

Furthermore,

$$\sup_{\boldsymbol{\xi}_i\in\boldsymbol{\Xi}_i}\frac{1}{T}\mathcal{L}^m_{iT}(\boldsymbol{\xi}_i,\widehat{\phi}_{\boldsymbol{\xi}\,NT})\xrightarrow{P}\sup_{\boldsymbol{\xi}_i\in\boldsymbol{\Xi}_i}\gamma_i(\boldsymbol{\xi}_i),\ \text{ as }\ T\to\infty,$$

which, by means of (B-77) and (B-78), is equivalent to

$$\frac{1}{T}\mathcal{L}^m_{iT}(\widehat{\boldsymbol{\xi}}_{iT},\widehat{\phi}_{\widehat{\boldsymbol{\xi}}_T\,NT})\xrightarrow{P}\gamma_i(\boldsymbol{\xi}_{i0}),\ \text{ as }\ T\to\infty,$$

where the curve is now estimated in $\widehat{\boldsymbol{\xi}}_T=(\widehat{\boldsymbol{\xi}}^{\mathrm{T}}_{1T}\dots\widehat{\boldsymbol{\xi}}^{\mathrm{T}}_{NT})^{\mathrm{T}}$. From (B-81), the term on the left hand side is such that

$$\frac{1}{T}\mathcal{L}^m_{iT}(\widehat{\boldsymbol{\xi}}_{iT},\widehat{\phi}_{\widehat{\boldsymbol{\xi}}_T\,NT})\xrightarrow{P}\gamma_i(\widehat{\boldsymbol{\xi}}_{iT}),\ \text{ as }\ T\to\infty,$$

thus

$$\gamma_i(\widehat{\boldsymbol{\xi}}_{iT})\xrightarrow{P}\gamma_i(\boldsymbol{\xi}_{i0}),\ \text{ as }\ T\to\infty.$$

Given assumptions I and S, we have consistency, for any $i=1,\dots,N$,

$$\widehat{\boldsymbol{\xi}}_{iT}\xrightarrow{P}\boldsymbol{\xi}_{i0},\ \text{ as }\ T\to\infty. \tag{B-82}$$

b) Given a consistent estimator of the curve $\widehat{\phi}_{\boldsymbol{\xi}\,NT}$, the estimated marginals' parameters, $\widehat{\boldsymbol{\xi}}_{1T},\dots,\widehat{\boldsymbol{\xi}}_{NT}$, have to satisfy (19). First order conditions and a Taylor series expansion around the true values $\boldsymbol{\xi}_{i0}$ give, for any $i=1,\dots,N$,

$$\mathbf{0}=\frac{1}{T}\frac{\partial\mathcal{L}^m_{iT}}{\partial\boldsymbol{\xi}_i}(\widehat{\boldsymbol{\xi}}_{iT},\widehat{\phi}_{\boldsymbol{\xi}\,NT})=\underbrace{\frac{1}{T}\frac{\partial\mathcal{L}^m_{iT}}{\partial\boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i0},\widehat{\phi}_{\boldsymbol{\xi}\,NT})}_{\mathcal{A}_{i\boldsymbol{\xi}_i T}}+\underbrace{\frac{1}{T}\frac{\partial^2\mathcal{L}^m_{iT}}{\partial\boldsymbol{\xi}_i\partial\boldsymbol{\xi}_i^{\mathrm{T}}}(\bar{\boldsymbol{\xi}}_i,\widehat{\phi}_{\boldsymbol{\xi}\,NT})}_{-\mathcal{B}_{i\bar{\boldsymbol{\xi}}_i T}}(\widehat{\boldsymbol{\xi}}_{iT}-\boldsymbol{\xi}_{i0}), \tag{B-83}$$

where $\bar{\boldsymbol{\xi}}_i$ is between $\widehat{\boldsymbol{\xi}}_{i\,T}$ and $\boldsymbol{\xi}_{i\,0}$. By rearranging (B-83), we get

$$\sqrt{T}(\widehat{\boldsymbol{\xi}}_{i\,T} - \boldsymbol{\xi}_{i\,0}) = \sqrt{T}\mathcal{A}_{i\boldsymbol{\xi}_i\,T}\left(\mathcal{B}_{i\bar{\boldsymbol{\xi}}_i\,T}\right)^{-1}. \tag{B-84}$$

Since both terms depend on the estimated curve, we cannot apply directly the Law of Large Numbers or the Central Limit Theorem.

By Lemma 3.d we have, for any $\boldsymbol{\xi}_i \in \Xi_i$, and any $i = 1, \ldots, N$,

$$\left|\left|\frac{1}{T}\frac{\partial}{\partial \boldsymbol{\xi}_i}\left(\mathcal{L}^m_{iT}(\boldsymbol{\xi}_i, \widehat{\phi}_{\boldsymbol{\xi}\,NT}) - \mathcal{L}^m_{iT}(\boldsymbol{\xi}_i, \phi_{\boldsymbol{\xi}})\right)\right|\right|_2 = \left|\left|\frac{1}{T}\frac{\partial}{\partial \boldsymbol{\xi}_i}\left(\frac{\partial \mathcal{L}^m_{iT}}{\partial \phi}(\boldsymbol{\xi}_i, \phi_{\boldsymbol{\xi}})(\widehat{\phi}_{\boldsymbol{\xi}\,NT} - \phi_{\boldsymbol{\xi}}) + r^{(2)}(\boldsymbol{\xi}_i)\right)\right|\right|_2,$$

where $\phi_{\boldsymbol{\xi}}$ is the least favorable curve. Using the last equality and a Taylor expansion around ϕ_0, we have

$$\begin{aligned}\left|\left|\mathcal{A}_{i\,\boldsymbol{\xi}_i\,T} - \frac{1}{T}\frac{\partial \mathcal{L}^m_{iT}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i\,0}, \phi_0)\right|\right|_2 \;\leq\; & \left|\left|\frac{1}{T}\frac{\partial}{\partial \boldsymbol{\xi}_i}\frac{\partial \mathcal{L}^m_{iT}}{\partial \phi}(\boldsymbol{\xi}_{i\,0}, \phi_0)(\widehat{\phi}_{\boldsymbol{\xi}_o\,NT} - \phi_0)\right|\right|_2 + \\ & + \left|\left|\frac{1}{T}\frac{\partial \mathcal{L}^m_{iT}}{\partial \phi}(\boldsymbol{\xi}_{i\,0}, \phi_0)(\widehat{\phi}'_{\boldsymbol{\xi}_o\,NT} - \phi'_0)\right|\right|_2 + \\ & + \left|\left|\frac{1}{T}\frac{\partial r^{(2)}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i\,0})\right|\right|_2,\end{aligned}$$

where the vectors $\widehat{\phi}'_{\boldsymbol{\xi}_o\,NT}$ and ϕ'_0 are defined in Theorem 3 and assumption L, respectively. By Lemma 3.a,b,d, we have, for any $i = 1, \ldots, N$,

$$\mathcal{A}_{i\,\boldsymbol{\xi}_i\,T} \xrightarrow{P} \frac{1}{T}\frac{\partial \mathcal{L}^m_{iT}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i\,0}, \phi_0), \;\; \text{as} \;\; T \to \infty.$$

By defining

$$\mathcal{A}^*_{i\,\boldsymbol{\xi}_i\,T} = \frac{1}{T}\frac{\partial \mathcal{L}^m_{NT}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i\,0}, \phi_0),$$

the Weak Law of Large Numbers and Lemma 1 in this paper, imply

$$\mathcal{A}^*_{i\,\boldsymbol{\xi}_i\,T} \xrightarrow{P} \mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}^m_{it}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i\,0}, \phi_0)\right] = \mathbf{0}, \;\; \text{as} \;\; T \to \infty. \tag{B-85}$$

A similar reasoning holds for $\mathcal{B}_{i\bar{\boldsymbol{\xi}}_i\,T}$. First, define

$$\mathcal{B}_{i\boldsymbol{\xi}_i\,T} = -\frac{1}{T}\frac{\partial^2 \mathcal{L}^m_{iT}}{\partial \boldsymbol{\xi}_i \partial \boldsymbol{\xi}_i^{\mathsf{T}}}(\boldsymbol{\xi}_{i\,0}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}).$$

Then, notice that, since $\left|\left|\bar{\boldsymbol{\xi}}_i - \boldsymbol{\xi}_{i\,0}\right|\right|_2 \leq \left|\left|\widehat{\boldsymbol{\xi}}_{i\,T} - \boldsymbol{\xi}_{i\,0}\right|\right|_2 = o_P(1)$ by part a) of this Theorem, and by using a Taylor series expansion in a neighborhood of $\boldsymbol{\xi}_{i\,0}$, we have

$$\left|\left|\mathcal{B}_{i\bar{\boldsymbol{\xi}}_i\,T} - \mathcal{B}_{i\boldsymbol{\xi}_i\,T}\right|\right|_2 = o_P(1).$$

From Lemma 3.c, we have, for any $\boldsymbol{\xi}_i \in \Xi_i$ and any $i = 1, \ldots, N$,

$$\sup_{\boldsymbol{\xi}_i \in \Xi_i}\left|\left|\frac{1}{T}\frac{\partial^2 \mathcal{L}^m_{iT}}{\partial \boldsymbol{\xi}_i \partial \boldsymbol{\xi}_i^{\mathsf{T}}}(\boldsymbol{\xi}_i, \widehat{\phi}_{\boldsymbol{\xi}\,NT}) - \frac{1}{T}\frac{\partial^2 \mathcal{L}^m_{iT}}{\partial \boldsymbol{\xi}_i \partial \boldsymbol{\xi}_i^{\mathsf{T}}}(\boldsymbol{\xi}_i, \phi_{\boldsymbol{\xi}})\right|\right|_2 = o_P(1), \;\; \text{as} \;\; T \to \infty,$$

which, computed in $\boldsymbol{\xi}_{i\,0}$, implies

$$\mathcal{B}_{i\,\boldsymbol{\xi}_i\,T} \xrightarrow{P} -\frac{1}{T}\frac{\partial^2 \mathcal{L}^m_{iT}}{\partial \boldsymbol{\xi}_i \partial \boldsymbol{\xi}_i^{\mathrm{T}}}(\boldsymbol{\xi}_{i\,0}, \phi_0), \text{ as } T \to \infty.$$

By defining

$$\mathcal{B}^*_{i\,\boldsymbol{\xi}_i\,T} = -\frac{1}{T}\frac{\partial^2 \mathcal{L}^m_{iT}}{\partial \boldsymbol{\xi}_i \partial \boldsymbol{\xi}_i^{\mathrm{T}}}(\boldsymbol{\xi}_{i\,0}, \phi_0),$$

the Weak Law of Large Numbers and Lemma 1 in this paper, imply

$$\mathcal{B}^*_{i\,\boldsymbol{\xi}_i\,T} \xrightarrow{P} -\mathrm{E}_0\left[\frac{\partial^2 \boldsymbol{\ell}^m_{it}}{\partial \boldsymbol{\xi}_i \partial \boldsymbol{\xi}_i^{\mathrm{T}}}(\boldsymbol{\xi}_{i\,0}, \phi_0)\right] = \mathcal{H}_{\boldsymbol{\xi}_{io}\boldsymbol{\xi}_{io}}, \text{ as } T \to \infty. \tag{B-86}$$

By combining (B-85) and (B-86) in (B-84), we have, for any $i = 1, \ldots, N$,

$$\left(\widehat{\boldsymbol{\xi}}_{i\,T} - \boldsymbol{\xi}_{i\,0}\right) \xrightarrow{P} \mathbf{0}, \text{ as } T \to \infty.$$

In order to study the asymptotic covariance matrix of the parameters of the marginals, we have to take into account the presence of the nuisance parameter, which depends on all parameters $\boldsymbol{\xi} = (\boldsymbol{\xi}_1^{\mathrm{T}} \ldots \boldsymbol{\xi}_N^{\mathrm{T}})^{\mathrm{T}}$. For this reason, by using (B-84) jointly for all marginals and Lemma 3, we can write

$$\sqrt{T}\left(\widehat{\boldsymbol{\xi}}_T - \boldsymbol{\xi}_0\right) = \sqrt{T}\begin{pmatrix} \mathcal{A}^*_{1\,\boldsymbol{\xi}_1\,T} \\ \vdots \\ \mathcal{A}^*_{N\,\boldsymbol{\xi}_N\,T} \end{pmatrix}\begin{pmatrix} \left(\mathcal{B}^*_{1\,\boldsymbol{\xi}_1\,T}\right)^{-1} & \ldots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \ldots & \left(\mathcal{B}^*_{N\,\boldsymbol{\xi}_N\,T}\right)^{-1} \end{pmatrix} = \sqrt{T}\mathcal{A}^*_{\boldsymbol{\xi}\,T}\left(\mathcal{B}^*_{\boldsymbol{\xi}\,T}\right)^{-1}.$$

By the Central Limit Theorem by Wooldridge and White (1988) and Slutsky's Theorem, we have

$$\sqrt{T}\left(\widehat{\boldsymbol{\xi}}_T - \boldsymbol{\xi}_0\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Omega}^*\right) \text{ as } T \to \infty.$$

where the asymptotic covariance matrix is

$$\boldsymbol{\Omega}^* = \left(\mathrm{E}_0\left[\mathcal{B}^*_{\boldsymbol{\xi}\,T}\right]\right)^{-1}\mathrm{Var}_0\left[\mathcal{A}^*_{\boldsymbol{\xi}\,T}\right]\left(\mathrm{E}_0\left[\mathcal{B}^*_{\boldsymbol{\xi}\,T}\right]\right)^{-1}.$$

First consider the case in which there is no correction due to the presence of a curve. Then

$$\mathrm{Var}_0\left[\mathcal{A}^*_{\boldsymbol{\xi}\,T}\right] = \mathrm{E}_0\left[\mathcal{A}^*_{\boldsymbol{\xi}\,T}\mathcal{A}^{*\mathrm{T}}_{\boldsymbol{\xi}\,T}\right] = \begin{pmatrix} \mathcal{I}_{\boldsymbol{\xi}_{1o}\,\boldsymbol{\xi}_{1o}} & \ldots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \ldots & \mathcal{I}_{\boldsymbol{\xi}_{No}\,\boldsymbol{\xi}_{No}} \end{pmatrix}, \tag{B-87}$$

where we used (B-85), Lemma 2, and we have defined

$$\mathcal{I}_{\boldsymbol{\xi}_{io}\,\boldsymbol{\xi}_{io}} = \mathrm{E}_0\left[\frac{\partial \boldsymbol{\ell}^m_{it}}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}_{i0}, \phi_0)\frac{\partial \boldsymbol{\ell}^m_{it}}{\partial \boldsymbol{\xi}_i^{\mathrm{T}}}(\boldsymbol{\xi}_{i0}, \phi_0)\right].$$

If we now correct for the presence of the curve we have to compute the correction using the least favorable direction. Thus, by adapting (16) in the paper to the multivariate case, (B-87) becomes

$$\mathbf{I}^*_{\boldsymbol{\xi}_o} = \mathrm{E}_0\left[\mathcal{A}^*_{\boldsymbol{\xi}T}\mathcal{A}^{*\mathrm{T}}_{\boldsymbol{\xi}T}\right] = \begin{pmatrix} \mathcal{I}_{\boldsymbol{\xi}_{1o}\boldsymbol{\xi}_{1o}} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{I}_{\boldsymbol{\xi}_{No}\boldsymbol{\xi}_{No}} \end{pmatrix} - \phi_0' \frac{1}{N}\mathrm{E}_0\left[\left(\sum_{i=1}^{N}\frac{\partial \boldsymbol{\ell}_{it}^m}{\partial \phi}(\boldsymbol{\xi}_0, \phi_0)\right)^2\right]\phi_0'^{\mathrm{T}}.$$

Now, by using ϕ_0' as defined in (18) in the paper and in Theorem 3, we have

$$\mathbf{I}^*_{\boldsymbol{\xi}_o} = \mathrm{E}_0\left[\mathcal{A}^*_{\boldsymbol{\xi}T}\mathcal{A}^{*\mathrm{T}}_{\boldsymbol{\xi}T}\right] = \begin{pmatrix} \mathcal{I}_{\boldsymbol{\xi}_{1o}\boldsymbol{\xi}_{1o}} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{I}_{\boldsymbol{\xi}_{No}\boldsymbol{\xi}_{No}} \end{pmatrix} - \bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_o}\bar{\boldsymbol{d}}^{\mathrm{T}}_{N\boldsymbol{\xi}_o} \otimes \frac{\bar{\imath}_{N\boldsymbol{\xi}_o}}{\bar{\jmath}^2_{N\boldsymbol{\xi}_o}}, \tag{B-88}$$

where $\bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_o}$ is defined in (B-70). More precisely, using (B-72) we can define

$$\bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_o} = \begin{pmatrix} \bar{\boldsymbol{d}}_{N\,\boldsymbol{\xi}_{1\,o}}(z_T) \\ \vdots \\ \bar{\boldsymbol{d}}_{N\,\boldsymbol{\xi}_{N\,o}}(z_T) \end{pmatrix},$$

and we have

$$\mathbf{I}^*_{\boldsymbol{\xi}_o} = \begin{pmatrix} \mathcal{I}_{\boldsymbol{\xi}_{1o}\boldsymbol{\xi}_{1o}} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{I}_{\boldsymbol{\xi}_{No}\boldsymbol{\xi}_{No}} \end{pmatrix} - \begin{pmatrix} \bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_{1\,o}}\bar{\boldsymbol{d}}^{\mathrm{T}}_{N\boldsymbol{\xi}_{1\,o}} & \dots & \bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_{1\,o}}\bar{\boldsymbol{d}}^{\mathrm{T}}_{N\boldsymbol{\xi}_{N,o}} \\ \vdots & \ddots & \vdots \\ \bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_{N\,o}}\bar{\boldsymbol{d}}^{\mathrm{T}}_{N\boldsymbol{\xi}_{1\,o}} & \dots & \bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_{N\,o}}\bar{\boldsymbol{d}}^{\mathrm{T}}_{N\boldsymbol{\xi}_{N,o}} \end{pmatrix} \otimes \frac{\bar{\imath}_{N\boldsymbol{\xi}_o}}{\bar{\jmath}^2_{N\boldsymbol{\xi}_o}},$$

Analogously, when correcting for the curve we can compute:

$$\mathbf{H}^*_{\boldsymbol{\xi}_o} = \mathrm{E}_0\left[\mathcal{B}^*_{\boldsymbol{\xi}T}\right] = \begin{pmatrix} \mathcal{H}_{\boldsymbol{\xi}_{1o}\boldsymbol{\xi}_{1o}} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{H}_{\boldsymbol{\xi}_{No}\boldsymbol{\xi}_{No}} \end{pmatrix} - \left(\bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_o}\bar{\boldsymbol{d}}^{\mathrm{T}}_{N\boldsymbol{\xi}_o}\right) \otimes \frac{1}{\bar{\jmath}_{N\boldsymbol{\xi}_o}}. \tag{B-89}$$

By combining (B-88) and (B-89) we have $\boldsymbol{\Omega}^* = \left(\mathbf{H}^*_{\boldsymbol{\xi}_o}\right)^{-1}\mathbf{I}^*_{\boldsymbol{\xi}_o}\left(\mathbf{H}^*_{\boldsymbol{\xi}_o}\right)^{-1}$, and

$$\sqrt{T}\left(\widehat{\boldsymbol{\xi}}_T - \boldsymbol{\xi}_0\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \left(\mathbf{H}^*_{\boldsymbol{\xi}_o}\right)^{-1}\mathbf{I}^*_{\boldsymbol{\xi}_o}\left(\mathbf{H}^*_{\boldsymbol{\xi}_o}\right)^{-1}\right) \text{ as } T \to \infty.$$

c) Let us consider each term of (B-88) separately. We see that the sums on the first term on the right hand side is $O(1)$. Moreover, the second term on the right hand side is the product of a term $\bar{\imath}_{N\boldsymbol{\xi}_o}/\bar{\jmath}^2_{N\boldsymbol{\xi}_o}$ which is bounded for any N provided assumptions C.2 and D hold, times the mixed derivatives $\bar{\boldsymbol{d}}_{N\boldsymbol{\xi}_o}$ which, by Theorem 3.b, are of order $O(N^{-1})$. Therefore, we have

$$\lim_{N\to\infty}\mathbf{I}^*_{\boldsymbol{\xi}_o} = \begin{pmatrix} \mathcal{I}_{\boldsymbol{\xi}_{1o}\boldsymbol{\xi}_{1o}} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{I}_{\boldsymbol{\xi}_{No}\boldsymbol{\xi}_{No}} \end{pmatrix} = \mathbf{I}_{\boldsymbol{\xi}_o}.$$

Analogously, provided assumptions C.2 holds, the same argument as before applies for (B-89), and we have

$$\lim_{N\to\infty}\mathbf{H}^*_{\boldsymbol{\xi}_o} = \begin{pmatrix} \mathcal{H}_{\boldsymbol{\xi}_{1o}\boldsymbol{\xi}_{1o}} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathcal{H}_{\boldsymbol{\xi}_{No}\boldsymbol{\xi}_{No}} \end{pmatrix} = \mathbf{H}_{\boldsymbol{\xi}_o}.$$

Thus, by applying the Central Limit Theorem by Wooldridge and White (1988) and Slutsky's Theorem, we have

$$\sqrt{T}\left(\widehat{\boldsymbol{\xi}}_T - \boldsymbol{\xi}_0\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \mathbf{H}_{\boldsymbol{\xi}_o}^{-1}\, \mathbf{I}_{\boldsymbol{\xi}_o}\, \mathbf{H}_{\boldsymbol{\xi}_o}^{-1}\right), \text{ as } N, T \to \infty.$$

This completes the proof. □

Proof of Theorem 5

We proceed as in Theorem 4.a. Define

$$\mathcal{L}_T^c(\widehat{\boldsymbol{\xi}}_T, \boldsymbol{\psi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}) = \sum_{t=1}^{T} \boldsymbol{\ell}_t^c(\widehat{\boldsymbol{\xi}}_T, \boldsymbol{\psi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}).$$

We prove consistency of $\widehat{\boldsymbol{\psi}}_T$, which is such that (see (20) in the paper),

$$\widehat{\boldsymbol{\psi}}_T = \arg\max_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \mathcal{L}_T^c(\widehat{\boldsymbol{\xi}}_T, \boldsymbol{\psi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}), \tag{B-90}$$

We define the function

$$\mu(\boldsymbol{\psi}) = \mathrm{E}_0\left[\boldsymbol{\ell}_t^c(\boldsymbol{\xi}_0, \boldsymbol{\psi}, \phi_{\boldsymbol{\xi}})\right].$$

From Lemma 1 and assumption H the true value of the parameters is such that

$$\boldsymbol{\psi}_0 = \arg\max_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \mu(\boldsymbol{\psi}). \tag{B-91}$$

Then, following the same argument as in the proof of Lemma 4 and using (B-13), we can use the Weak Law of Large Numbers by McLeish (1975) and Lemma 2.1 of White and Domowitz (1984), which imply that, for any $\boldsymbol{\psi} \in \boldsymbol{\Psi}$,

$$\frac{1}{T}\mathcal{L}_T^c(\boldsymbol{\xi}_0, \boldsymbol{\psi}, \phi_{\boldsymbol{\xi}}) \xrightarrow{P} \mu(\boldsymbol{\psi}), \text{ as } T \to \infty. \tag{B-92}$$

Similarly to what proved above we can use Corollary 1 and (B-82) to prove that, for any $\boldsymbol{\psi} \in \boldsymbol{\Psi}$,

$$\frac{1}{T}\left|\mathcal{L}_T^c(\widehat{\boldsymbol{\xi}}_T, \boldsymbol{\psi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}) - \mathcal{L}_T^c(\boldsymbol{\xi}_0, \boldsymbol{\psi}, \phi_{\boldsymbol{\xi}})\right| \xrightarrow{P} 0, \text{ as } T \to \infty. \tag{B-93}$$

Therefore, by combining (B-92) and (B-93), we have, for any $\boldsymbol{\psi} \in \boldsymbol{\Psi}$,

$$\frac{1}{T}\mathcal{L}_T^c(\widehat{\boldsymbol{\xi}}_T, \boldsymbol{\psi}, \widehat{\phi}_{\boldsymbol{\xi}\,NT}) \xrightarrow{P} \mu(\boldsymbol{\psi}), \text{ as } T \to \infty. \tag{B-94}$$

Furthermore, by means of (B-90) and (B-91), we get

$$\frac{1}{T}\mathcal{L}_T^c(\widehat{\boldsymbol{\xi}}_T, \widehat{\boldsymbol{\psi}}_T, \widehat{\phi}_{\widehat{\boldsymbol{\xi}}_T\,NT}) \xrightarrow{P} \mu(\boldsymbol{\psi}_0), \text{ as } T \to \infty.$$

From (B-94), the term on the left hand side is such that

$$\frac{1}{T}\mathcal{L}_T^c(\widehat{\boldsymbol{\xi}}_T, \widehat{\boldsymbol{\psi}}_T, \widehat{\phi}_{\widehat{\boldsymbol{\xi}}_T\,NT}) \xrightarrow{P} \mu(\widehat{\boldsymbol{\psi}}_T), \text{ as } T \to \infty,$$

thus

$$\mu(\widehat{\boldsymbol{\psi}}_T) \xrightarrow{P} \mu(\boldsymbol{\psi}_0), \text{ as } T \to \infty.$$

Given assumptions I and S, we have consistency

$$\widehat{\boldsymbol{\psi}}_T \xrightarrow{P} \boldsymbol{\psi}_0, \text{ as } T \to \infty.$$

This completes the proof. $\square$

Appendix C - Data Description and Detailed Estimation Results

Table 11: S&P100 constituents

Ticker	Name	Sector
AA	Alcoa Inc	Materials
AAPL	Apple Inc.	Information Technology
ABT	Abbott Labs	Health Care
AEP	American Electric Power	Utilities
ALL	Allstate Corp.	Financials
AMGN	Amgen	Health Care
AMZN	Amazon Corp.	Consumer Discretionary
AVP	Avon Products	Consumer Staples
AXP	American Express	Financials
BA	Boeing Company	Industrials
BAC	Bank of America Corp.	Financials
BAX	Baxter International Inc.	Health Care
BHI	Baker Hughes	Energy
BK	Bank of New York Mellon Corp.	Financials
BMY	Bristol-Myers Squibb	Health Care
BNI	Burlington Northern Santa Fe C	Industrials
CAT	Caterpillar Inc.	Industrials
C	Citigroup Inc.	Financials
CL	Colgate-Palmolive	Consumer Staples
CMCSA	Comcast Corp.	Consumer Discretionary
COF	Capital One Financial	Financials
COST	Costco Co.	Consumer Staples
CPB	Campbell Soup	Consumer Staples
CSCO	Cisco Systems	Information Technology
CVS	CVS Caremark Corp.	Consumer Staples
CVX	Chevron Corp.	Energy
DD	Du Pont (E.I.)	Materials
DELL	Dell Inc.	Information Technology
DIS	Walt Disney Co.	Consumer Discretionary
DOW	Dow Chemical	Materials
DVN	Devon Energy Corp.	Energy
EMC	EMC Corp.	Information Technology
ETR	Entergy Corp.	Utilities
EXC	Exelon Corp.	Utilities
FDX	FedEx Corporation	Industrials
F	Ford Motor	Consumer Discretionary
GD	General Dynamics	Industrials
GE	General Electric	Industrials
GILD	Gilead Sciences	Health Care
GS	Goldman Sachs Group	Financials
HAL	Halliburton Co.	Energy
HD	Home Depot	Consumer Discretionary
HNZ	Heinz (H.J.)	Consumer Staples
HON	Honeywell Int'l Inc.	Industrials
HPQ	Hewlett-Packard	Information Technology
(cont.)		

(cont.)

IBM	International Bus. Machines	Information Technology
INTC	Intel Corp.	Information Technology
JNJ	Johnson & Johnson	Health Care
JPM	JPMorgan Chase & Co.	Financials
KO	Coca Cola Co.	Consumer Staples
LMT	Lockheed Martin Corp.	Industrials
LOW	Lowe's Cos.	Consumer Discretionary
MCD	McDonald's Corp.	Consumer Discretionary
MDT	Medtronic Inc.	Health Care
MMM	3M Company	Industrials
MO	Altria Group, Inc.	Consumer Staples
MRK	Merck & Co.	Health Care
MSFT	Microsoft Corp.	Information Technology
MS	Morgan Stanley	Financials
NKE	NIKE Inc.	Consumer Discretionary
NSC	Norfolk Southern Corp.	Industrials
ORCL	Oracle Corp.	Information Technology
OXY	Occidental Petroleum	Energy
PEP	PepsiCo Inc.	Consumer Staples
PFE	Pfizer, Inc.	Health Care
PG	Procter & Gamble	Consumer Staples
QCOM	QUALCOMM Inc.	Information Technology
RF	Regions Financial Corp.	Financials
SGP	Schering-Plough	Health Care
SLB	Schlumberger Ltd.	Energy
SLE	Sara Lee Corp.	Consumer Staples
SO	Southern Co.	Utilities
S	Sprint Nextel Corp.	Telecommunications Services
T	AT&T Inc.	Telecommunications Services
TGT	Target Corp.	Consumer Discretionary
TWX	Time Warner Inc.	Consumer Discretionary
TXN	Texas Instruments	Information Technology
TYC	Tyco International	Industrials
UNH	UnitedHealth Group Inc.	Health Care
UPS	United Parcel Service	Industrials
USB	U.S. Bancorp	Financials
UTX	United Technologies	Industrials
VZ	Verizon Communications	Telecommunications Services
WAG	Walgreen Co.	Consumer Staples
WFC	Wells Fargo	Financials
WMB	Williams Cos.	Energy
WMT	Wal-Mart Stores	Consumer Staples
WY	Weyerhaeuser Corp.	Materials
XOM	Exxon Mobil Corp.	Energy
XRX	Xerox Corp.	Information Technology

Table 12: S&P100 Parameter Estimates

	SPVMEM						MEM					
	a_i	α_i	γ_i	β_i	ν_i	π_i	a_i	α_i	γ_i	β_i	ν_i	π_i
AA	0.35 (0.041)	0.28 (0.020)	0.08 (0.013)	0.63 (0.019)	0.27 (0.000)	0.94 (0.029)	0.16 (0.036)	0.28 (0.034)	0.08 (0.021)	0.65 (0.029)	0.28 (0.027)	0.97 (0.045)
AAPL	0.55 (0.037)	0.34 (0.018)	0.12 (0.012)	0.53 (0.018)	0.48 (0.000)	0.93 (0.027)	0.34 (0.080)	0.34 (0.054)	0.12 (0.035)	0.55 (0.051)	0.51 (0.037)	0.95 (0.076)
ABT	0.10 (0.013)	0.28 (0.016)	0.09 (0.011)	0.65 (0.013)	0.34 (0.000)	0.98 (0.021)	0.07 (0.020)	0.29 (0.037)	0.09 (0.027)	0.65 (0.030)	0.34 (0.030)	0.98 (0.049)
AEP	0.10 (0.012)	0.26 (0.013)	0.09 (0.010)	0.67 (0.012)	0.38 (0.000)	0.98 (0.018)	0.07 (0.019)	0.26 (0.029)	0.10 (0.023)	0.67 (0.027)	0.38 (0.026)	0.99 (0.041)
ALL	0.09 (0.011)	0.29 (0.015)	0.10 (0.012)	0.65 (0.014)	0.34 (0.000)	0.98 (0.021)	0.05 (0.016)	0.29 (0.034)	0.10 (0.028)	0.65 (0.030)	0.34 (0.028)	0.99 (0.048)
AMGN	0.17 (0.025)	0.38 (0.024)	0.07 (0.018)	0.54 (0.025)	0.28 (0.000)	0.95 (0.036)	0.13 (0.025)	0.38 (0.034)	0.08 (0.025)	0.53 (0.032)	0.29 (0.023)	0.96 (0.049)
AMZN	0.43 (0.043)	0.31 (0.016)	0.10 (0.014)	0.59 (0.016)	0.32 (0.000)	0.95 (0.023)	0.17 (0.053)	0.31 (0.036)	0.10 (0.032)	0.62 (0.030)	0.35 (0.034)	0.98 (0.049)
AVP	0.13 (0.014)	0.41 (0.013)	0.06 (0.009)	0.55 (0.013)	0.67 (0.000)	0.98 (0.019)	0.30 (0.080)	0.40 (0.052)	0.08 (0.043)	0.48 (0.061)	0.64 (0.041)	0.92 (0.083)
AXP	0.10 (0.014)	0.35 (0.016)	0.12 (0.013)	0.57 (0.016)	0.30 (0.000)	0.98 (0.024)	0.04 (0.013)	0.36 (0.033)	0.11 (0.025)	0.58 (0.030)	0.30 (0.027)	1.00 (0.047)
BA	0.11 (0.018)	0.24 (0.020)	0.11 (0.014)	0.68 (0.018)	0.24 (0.000)	0.98 (0.028)	0.06 (0.017)	0.24 (0.030)	0.11 (0.021)	0.69 (0.026)	0.25 (0.024)	0.99 (0.041)
BAC	0.12 (0.009)	0.37 (0.020)	0.14 (0.013)	0.52 (0.017)	0.42 (0.000)	0.97 (0.027)	0.05 (0.014)	0.37 (0.058)	0.14 (0.038)	0.54 (0.048)	0.44 (0.043)	0.99 (0.077)
BAX	0.07 (0.008)	0.31 (0.011)	0.09 (0.009)	0.64 (0.009)	0.65 (0.000)	1.00 (0.015)	0.08 (0.035)	0.31 (0.063)	0.09 (0.047)	0.63 (0.049)	0.64 (0.048)	0.99 (0.083)
BHI	0.20 (0.045)	0.27 (0.023)	0.06 (0.014)	0.67 (0.022)	0.20 (0.000)	0.98 (0.033)	0.14 (0.036)	0.26 (0.025)	0.07 (0.015)	0.68 (0.024)	0.21 (0.022)	0.98 (0.036)
BK	0.12 (0.014)	0.32 (0.009)	0.07 (0.012)	0.63 (0.006)	0.59 (0.000)	0.99 (0.013)	0.08 (0.049)	0.32 (0.051)	0.08 (0.066)	0.63 (0.033)	0.58 (0.057)	1.00 (0.069)
BMY	0.16 (0.018)	0.26 (0.011)	0.02 (0.009)	0.70 (0.011)	0.46 (0.000)	0.97 (0.016)	0.08 (0.027)	0.26 (0.031)	0.02 (0.026)	0.71 (0.028)	0.46 (0.028)	0.98 (0.043)
BNI	0.22 (0.028)	0.34 (0.019)	0.08 (0.015)	0.58 (0.019)	0.28 (0.000)	0.96 (0.028)	0.19 (0.041)	0.33 (0.036)	0.09 (0.029)	0.58 (0.036)	0.27 (0.026)	0.96 (0.053)
C	0.10 (0.015)	0.33 (0.021)	0.16 (0.015)	0.56 (0.017)	0.25 (0.000)	0.98 (0.028)	0.03 (0.012)	0.34 (0.033)	0.16 (0.024)	0.58 (0.025)	0.25 (0.025)	1.00 (0.043)
CAT	0.19 (0.029)	0.35 (0.023)	0.07 (0.016)	0.57 (0.023)	0.21 (0.000)	0.96 (0.033)	0.12 (0.028)	0.35 (0.034)	0.08 (0.024)	0.58 (0.032)	0.21 (0.028)	0.97 (0.048)
CL	0.07 (0.009)	0.35 (0.016)	0.05 (0.010)	0.61 (0.014)	0.39 (0.000)	0.98 (0.022)	0.07 (0.026)	0.36 (0.054)	0.06 (0.033)	0.59 (0.048)	0.39 (0.037)	0.98 (0.074)
CMCSA	0.13 (0.021)	0.33 (0.020)	0.08 (0.013)	0.60 (0.019)	0.27 (0.000)	0.97 (0.028)	0.09 (0.031)	0.33 (0.042)	0.08 (0.027)	0.60 (0.041)	0.28 (0.034)	0.98 (0.060)
COF	0.20 (0.030)	0.37 (0.019)	0.13 (0.014)	0.55 (0.018)	0.27 (0.000)	0.98 (0.027)	0.08 (0.025)	0.37 (0.033)	0.13 (0.025)	0.56 (0.030)	0.27 (0.025)	1.00 (0.047)
COST	0.23 (0.031)	0.37 (0.022)	0.05 (0.015)	0.55 (0.023)	0.26 (0.000)	0.94 (0.032)	0.13 (0.047)	0.38 (0.052)	0.04 (0.038)	0.57 (0.051)	0.26 (0.039)	0.96 (0.075)
CPB	0.09 (0.006)	0.27 (0.013)	0.04 (0.010)	0.68 (0.011)	0.58 (0.000)	0.98 (0.018)	0.09 (0.031)	0.29 (0.077)	0.05 (0.063)	0.66 (0.069)	0.56 (0.065)	0.97 (0.109)
CSCO	0.31 (0.040)	0.49 (0.022)	0.10 (0.018)	0.40 (0.021)	0.25 (0.000)	0.94 (0.032)	0.15 (0.040)	0.50 (0.043)	0.10 (0.036)	0.42 (0.036)	0.26 (0.034)	0.97 (0.059)
CVS	0.14 (0.015)	0.24 (0.010)	0.00 (0.008)	0.74 (0.011)	0.55 (0.000)	0.98 (0.015)	0.10 (0.031)	0.24 (0.033)	0.01 (0.024)	0.74 (0.032)	0.56 (0.032)	0.98 (0.047)
CVX	0.08 (0.015)	0.35 (0.022)	0.08 (0.015)	0.59 (0.020)	0.22 (0.000)	0.98 (0.031)	0.12 (0.027)	0.34 (0.035)	0.09 (0.025)	0.56 (0.034)	0.22 (0.027)	0.95 (0.050)
DD	0.17 (0.022)	0.30 (0.018)	0.10 (0.015)	0.60 (0.018)	0.25 (0.000)	0.96 (0.026)	0.09 (0.027)	0.30 (0.037)	0.10 (0.031)	0.62 (0.034)	0.26 (0.033)	0.97 (0.052)
DELL	0.21 (0.036)	0.39 (0.027)	0.09 (0.018)	0.52 (0.026)	0.19 (0.000)	0.95 (0.039)	0.09 (0.028)	0.40 (0.037)	0.08 (0.027)	0.54 (0.035)	0.19 (0.029)	0.98 (0.053)
DIS	0.11 (0.010)	0.27 (0.013)	0.09 (0.011)	0.66 (0.013)	0.38 (0.000)	0.98 (0.019)	0.06 (0.019)	0.27 (0.044)	0.09 (0.037)	0.66 (0.041)	0.39 (0.043)	0.99 (0.063)
DOW	0.17 (0.017)	0.25 (0.012)	0.11 (0.011)	0.67 (0.010)	0.37 (0.000)	0.97 (0.016)	0.08 (0.032)	0.26 (0.039)	0.10 (0.036)	0.68 (0.029)	0.39 (0.040)	0.99 (0.052)

	SPVMEM						MEM					
	a_i	α_i	γ_i	β_i	ν_i	π_i	a_i	α_i	γ_i	β_i	ν_i	π_i
DVN	0.11 (0.020)	0.26 (0.016)	0.05 (0.011)	0.70 (0.014)	0.40 (0.000)	0.98 (0.022)	0.17 (0.058)	0.24 (0.043)	0.07 (0.027)	0.69 (0.040)	0.41 (0.036)	0.97 (0.060)
EMC	0.22 (0.032)	0.25 (0.015)	0.09 (0.011)	0.68 (0.013)	0.35 (0.000)	0.98 (0.021)	0.11 (0.031)	0.26 (0.028)	0.09 (0.022)	0.69 (0.024)	0.36 (0.025)	0.99 (0.038)
ETR	0.09 (0.011)	0.30 (0.017)	0.10 (0.014)	0.63 (0.015)	0.27 (0.000)	0.98 (0.024)	0.07 (0.016)	0.29 (0.031)	0.10 (0.025)	0.63 (0.028)	0.28 (0.025)	0.98 (0.044)
EXC	0.13 (0.016)	0.33 (0.012)	0.08 (0.013)	0.60 (0.012)	0.34 (0.000)	0.97 (0.019)	0.10 (0.035)	0.32 (0.041)	0.09 (0.039)	0.61 (0.037)	0.33 (0.041)	0.97 (0.059)
F	0.35 (0.032)	0.34 (0.010)	0.07 (0.009)	0.60 (0.009)	0.67 (0.000)	0.97 (0.014)	0.20 (0.087)	0.34 (0.054)	0.07 (0.047)	0.61 (0.042)	0.64 (0.046)	0.98 (0.072)
FDX	0.15 (0.026)	0.31 (0.015)	0.05 (0.014)	0.63 (0.017)	0.25 (0.000)	0.96 (0.024)	0.06 (0.028)	0.30 (0.029)	0.05 (0.028)	0.65 (0.028)	0.25 (0.033)	0.98 (0.043)
GD	0.14 (0.021)	0.33 (0.018)	0.07 (0.014)	0.59 (0.018)	0.29 (0.000)	0.96 (0.026)	0.08 (0.024)	0.33 (0.034)	0.07 (0.028)	0.61 (0.031)	0.29 (0.030)	0.98 (0.048)
GE	0.10 (0.014)	0.34 (0.020)	0.12 (0.016)	0.56 (0.018)	0.26 (0.000)	0.96 (0.028)	0.03 (0.014)	0.35 (0.043)	0.11 (0.035)	0.59 (0.038)	0.27 (0.035)	0.99 (0.060)
GILD	0.24 (0.050)	0.41 (0.023)	0.06 (0.018)	0.53 (0.024)	0.20 (0.000)	0.97 (0.034)	0.17 (0.058)	0.41 (0.035)	0.07 (0.028)	0.53 (0.034)	0.20 (0.031)	0.97 (0.051)
GS	0.16 (0.027)	0.43 (0.023)	0.13 (0.018)	0.47 (0.014)	0.22 (0.000)	0.97 (0.028)	0.07 (0.033)	0.44 (0.045)	0.13 (0.036)	0.48 (0.027)	0.23 (0.038)	0.99 (0.055)
HAL	0.23 (0.034)	0.25 (0.014)	0.10 (0.012)	0.68 (0.010)	0.40 (0.000)	0.98 (0.018)	0.13 (0.089)	0.25 (0.059)	0.10 (0.052)	0.69 (0.042)	0.41 (0.058)	0.99 (0.077)
HD	0.17 (0.026)	0.30 (0.018)	0.09 (0.013)	0.62 (0.018)	0.26 (0.000)	0.96 (0.026)	0.07 (0.024)	0.30 (0.033)	0.10 (0.024)	0.64 (0.029)	0.27 (0.028)	0.98 (0.045)
HNZ	0.04 (0.006)	0.19 (0.008)	0.06 (0.006)	0.77 (0.007)	0.56 (0.000)	0.99 (0.011)	0.04 (0.018)	0.19 (0.031)	0.07 (0.025)	0.76 (0.028)	0.56 (0.034)	0.99 (0.043)
HON	0.14 (0.021)	0.33 (0.015)	0.09 (0.011)	0.60 (0.013)	0.33 (0.000)	0.98 (0.021)	0.08 (0.032)	0.33 (0.038)	0.09 (0.028)	0.61 (0.031)	0.33 (0.034)	0.99 (0.051)
HPQ	0.11 (0.015)	0.25 (0.014)	0.12 (0.011)	0.68 (0.012)	0.37 (0.000)	0.99 (0.019)	0.06 (0.020)	0.26 (0.030)	0.11 (0.023)	0.68 (0.026)	0.38 (0.026)	0.99 (0.042)
IBM	0.10 (0.015)	0.30 (0.021)	0.14 (0.015)	0.59 (0.021)	0.22 (0.000)	0.96 (0.031)	0.04 (0.013)	0.31 (0.032)	0.14 (0.024)	0.60 (0.028)	0.24 (0.029)	0.98 (0.044)
INTC	0.28 (0.043)	0.43 (0.022)	0.13 (0.019)	0.45 (0.025)	0.20 (0.000)	0.94 (0.035)	0.14 (0.044)	0.43 (0.046)	0.13 (0.039)	0.47 (0.046)	0.21 (0.040)	0.97 (0.068)
JNJ	0.04 (0.006)	0.26 (0.016)	0.13 (0.011)	0.66 (0.013)	0.37 (0.000)	0.99 (0.021)	0.02 (0.009)	0.27 (0.040)	0.12 (0.027)	0.66 (0.031)	0.38 (0.033)	1.00 (0.052)
JPM	0.14 (0.016)	0.37 (0.020)	0.14 (0.014)	0.54 (0.017)	0.30 (0.000)	0.98 (0.027)	0.05 (0.016)	0.38 (0.039)	0.13 (0.027)	0.55 (0.033)	0.31 (0.028)	1.00 (0.052)
KO	0.09 (0.009)	0.33 (0.012)	0.07 (0.011)	0.61 (0.014)	0.36 (0.000)	0.97 (0.019)	0.04 (0.016)	0.33 (0.041)	0.06 (0.038)	0.62 (0.043)	0.38 (0.048)	0.99 (0.063)
LMT	0.14 (0.016)	0.31 (0.018)	0.09 (0.012)	0.61 (0.016)	0.35 (0.000)	0.96 (0.025)	0.06 (0.020)	0.32 (0.038)	0.09 (0.028)	0.63 (0.032)	0.37 (0.029)	0.99 (0.051)
LOW	0.27 (0.038)	0.31 (0.020)	0.10 (0.014)	0.59 (0.021)	0.25 (0.000)	0.95 (0.030)	0.11 (0.034)	0.32 (0.032)	0.09 (0.024)	0.61 (0.031)	0.25 (0.029)	0.98 (0.047)
MCD	0.08 (0.009)	0.23 (0.011)	0.05 (0.009)	0.73 (0.010)	0.43 (0.000)	0.98 (0.016)	0.04 (0.017)	0.23 (0.037)	0.05 (0.029)	0.74 (0.030)	0.43 (0.037)	0.99 (0.050)
MDT	0.08 (0.012)	0.25 (0.012)	0.09 (0.007)	0.69 (0.011)	0.60 (0.000)	0.98 (0.017)	0.06 (0.026)	0.25 (0.037)	0.09 (0.023)	0.69 (0.034)	0.61 (0.034)	0.98 (0.051)
MMM	0.11 (0.017)	0.37 (0.021)	0.10 (0.016)	0.54 (0.022)	0.23 (0.000)	0.96 (0.031)	0.09 (0.020)	0.37 (0.037)	0.11 (0.026)	0.54 (0.036)	0.23 (0.030)	0.96 (0.053)
MO	0.13 (0.010)	0.32 (0.010)	0.07 (0.009)	0.61 (0.009)	0.70 (0.000)	0.97 (0.014)	0.10 (0.049)	0.33 (0.067)	0.08 (0.059)	0.61 (0.059)	0.67 (0.059)	0.97 (0.094)
MRK	0.17 (0.013)	0.34 (0.014)	0.07 (0.009)	0.59 (0.011)	1.00 (0.000)	0.97 (0.018)	0.22 (0.084)	0.34 (0.089)	0.10 (0.059)	0.55 (0.081)	0.89 (0.057)	0.94 (0.124)
MS	0.30 (0.043)	0.39 (0.021)	0.16 (0.016)	0.49 (0.019)	0.24 (0.000)	0.96 (0.029)	0.12 (0.056)	0.40 (0.052)	0.16 (0.040)	0.51 (0.044)	0.24 (0.042)	0.99 (0.071)
MSFT	0.16 (0.019)	0.37 (0.019)	0.09 (0.016)	0.52 (0.021)	0.23 (0.000)	0.94 (0.029)	0.05 (0.015)	0.38 (0.029)	0.08 (0.026)	0.56 (0.029)	0.24 (0.030)	0.98 (0.043)
NKE	0.24 (0.025)	0.28 (0.015)	0.10 (0.013)	0.61 (0.017)	0.35 (0.000)	0.94 (0.024)	0.08 (0.029)	0.26 (0.049)	0.10 (0.042)	0.67 (0.045)	0.36 (0.044)	0.98 (0.069)

	SPVMEM						MEM					
	a_i	α_i	γ_i	β_i	ν_i	π_i	a_i	α_i	γ_i	β_i	ν_i	π_i
NSC	0.22 (0.032)	0.28 (0.014)	0.06 (0.013)	0.66 (0.015)	0.28 (0.000)	0.97 (0.022)	0.12 (0.045)	0.28 (0.030)	0.06 (0.027)	0.68 (0.029)	0.29 (0.029)	0.98 (0.044)
ORCL	0.27 (0.037)	0.45 (0.023)	0.08 (0.018)	0.47 (0.022)	0.24 (0.000)	0.96 (0.034)	0.15 (0.042)	0.45 (0.046)	0.09 (0.036)	0.48 (0.041)	0.25 (0.032)	0.98 (0.064)
OXY	0.13 (0.021)	0.25 (0.020)	0.08 (0.014)	0.68 (0.019)	0.25 (0.000)	0.98 (0.028)	0.14 (0.030)	0.25 (0.028)	0.09 (0.020)	0.68 (0.027)	0.26 (0.022)	0.97 (0.040)
PEP	0.06 (0.007)	0.23 (0.014)	0.09 (0.009)	0.70 (0.012)	0.42 (0.000)	0.98 (0.019)	0.03 (0.013)	0.23 (0.042)	0.09 (0.030)	0.71 (0.034)	0.43 (0.040)	0.99 (0.056)
PFE	0.11 (0.016)	0.42 (0.007)	−0.03 (0.009)	0.58 (0.010)	0.70 (0.000)	0.98 (0.013)	0.08 (0.069)	0.41 (0.055)	−0.02 (0.062)	0.58 (0.062)	0.66 (0.075)	0.98 (0.089)
PG	0.07 (0.008)	0.32 (0.018)	0.08 (0.013)	0.61 (0.016)	0.33 (0.000)	0.97 (0.025)	0.06 (0.017)	0.33 (0.050)	0.08 (0.037)	0.60 (0.045)	0.33 (0.042)	0.97 (0.070)
QCOM	0.21 (0.036)	0.29 (0.018)	0.13 (0.015)	0.61 (0.016)	0.27 (0.000)	0.97 (0.025)	0.15 (0.052)	0.30 (0.040)	0.14 (0.034)	0.60 (0.031)	0.28 (0.037)	0.97 (0.054)
RF	0.05 (0.006)	0.26 (0.011)	0.08 (0.009)	0.70 (0.010)	0.55 (0.000)	1.00 (0.016)	0.06 (0.016)	0.26 (0.029)	0.08 (0.025)	0.69 (0.026)	0.53 (0.024)	0.99 (0.041)
S	0.29 (0.028)	0.29 (0.011)	0.12 (0.009)	0.63 (0.008)	0.75 (0.000)	0.98 (0.014)	0.10 (0.043)	0.30 (0.034)	0.11 (0.029)	0.65 (0.025)	0.77 (0.029)	1.00 (0.044)
SGP	0.24 (0.017)	0.27 (0.010)	0.00 (0.009)	0.70 (0.008)	0.90 (0.000)	0.97 (0.013)	0.08 (0.040)	0.27 (0.047)	0.00 (0.041)	0.72 (0.033)	0.96 (0.040)	0.99 (0.061)
SLB	0.24 (0.043)	0.29 (0.023)	0.07 (0.016)	0.64 (0.023)	0.19 (0.000)	0.96 (0.034)	0.15 (0.031)	0.28 (0.024)	0.08 (0.016)	0.65 (0.023)	0.19 (0.021)	0.97 (0.034)
SLE	0.12 (0.010)	0.28 (0.008)	0.02 (0.007)	0.69 (0.008)	0.75 (0.000)	0.98 (0.012)	0.10 (0.047)	0.30 (0.054)	0.02 (0.046)	0.67 (0.051)	0.73 (0.047)	0.98 (0.078)
SO	0.04 (0.008)	0.27 (0.013)	0.07 (0.011)	0.68 (0.009)	0.36 (0.000)	0.99 (0.017)	0.03 (0.015)	0.28 (0.034)	0.07 (0.029)	0.68 (0.025)	0.37 (0.034)	0.99 (0.045)
T	0.15 (0.016)	0.29 (0.012)	0.08 (0.011)	0.65 (0.010)	0.46 (0.000)	0.98 (0.017)	0.06 (0.022)	0.30 (0.035)	0.08 (0.035)	0.66 (0.030)	0.47 (0.032)	1.00 (0.049)
TGT	0.17 (0.024)	0.26 (0.016)	0.10 (0.013)	0.67 (0.015)	0.29 (0.000)	0.97 (0.023)	0.07 (0.028)	0.27 (0.031)	0.09 (0.027)	0.68 (0.029)	0.30 (0.030)	0.99 (0.045)
TWX	0.11 (0.017)	0.29 (0.015)	0.07 (0.009)	0.66 (0.015)	0.37 (0.000)	0.98 (0.021)	0.06 (0.020)	0.29 (0.038)	0.07 (0.025)	0.66 (0.036)	0.37 (0.033)	0.99 (0.054)
TXN	0.10 (0.027)	0.24 (0.016)	0.11 (0.013)	0.70 (0.015)	0.21 (0.000)	0.99 (0.023)	0.07 (0.029)	0.24 (0.029)	0.11 (0.024)	0.69 (0.026)	0.22 (0.031)	0.99 (0.040)
TYC	0.12 (0.016)	0.34 (0.014)	0.09 (0.012)	0.61 (0.012)	0.39 (0.000)	0.99 (0.020)	0.08 (0.028)	0.34 (0.035)	0.09 (0.030)	0.61 (0.029)	0.39 (0.027)	0.99 (0.048)
UNH	0.14 (0.021)	0.33 (0.013)	0.06 (0.010)	0.61 (0.014)	0.41 (0.000)	0.98 (0.020)	0.13 (0.043)	0.34 (0.035)	0.06 (0.026)	0.60 (0.036)	0.41 (0.032)	0.97 (0.052)
UPS	0.08 (0.008)	0.28 (0.013)	0.12 (0.010)	0.63 (0.013)	0.40 (0.000)	0.97 (0.019)	0.09 (0.030)	0.28 (0.047)	0.12 (0.036)	0.62 (0.046)	0.41 (0.041)	0.96 (0.068)
USB	0.10 (0.005)	0.28 (0.012)	0.11 (0.011)	0.65 (0.011)	0.44 (0.000)	0.99 (0.018)	0.05 (0.009)	0.28 (0.040)	0.11 (0.036)	0.66 (0.035)	0.46 (0.035)	1.00 (0.056)
UTX	0.08 (0.015)	0.28 (0.018)	0.13 (0.014)	0.64 (0.016)	0.28 (0.000)	0.98 (0.025)	0.07 (0.019)	0.28 (0.031)	0.14 (0.025)	0.64 (0.029)	0.28 (0.026)	0.98 (0.044)
VZ	0.12 (0.015)	0.29 (0.018)	0.09 (0.013)	0.63 (0.015)	0.29 (0.000)	0.97 (0.024)	0.05 (0.019)	0.31 (0.041)	0.08 (0.031)	0.64 (0.034)	0.29 (0.035)	0.99 (0.056)
WAG	0.13 (0.018)	0.24 (0.016)	0.06 (0.011)	0.70 (0.014)	0.33 (0.000)	0.97 (0.022)	0.07 (0.025)	0.23 (0.037)	0.07 (0.027)	0.71 (0.031)	0.34 (0.031)	0.98 (0.050)
WFC	0.11 (0.011)	0.32 (0.017)	0.12 (0.014)	0.59 (0.017)	0.29 (0.000)	0.97 (0.025)	0.05 (0.012)	0.32 (0.032)	0.11 (0.027)	0.61 (0.029)	0.31 (0.026)	0.99 (0.046)
WMB	0.12 (0.019)	0.25 (0.008)	0.06 (0.008)	0.71 (0.006)	0.66 (0.000)	0.99 (0.011)	0.09 (0.058)	0.25 (0.031)	0.08 (0.030)	0.71 (0.023)	0.67 (0.034)	1.00 (0.041)
WMT	0.10 (0.016)	0.30 (0.019)	0.09 (0.013)	0.62 (0.019)	0.26 (0.000)	0.97 (0.028)	0.05 (0.014)	0.31 (0.031)	0.09 (0.023)	0.63 (0.030)	0.27 (0.029)	0.98 (0.044)
WY	0.20 (0.029)	0.33 (0.019)	0.10 (0.016)	0.58 (0.020)	0.23 (0.000)	0.96 (0.029)	0.17 (0.039)	0.34 (0.034)	0.10 (0.027)	0.57 (0.035)	0.23 (0.026)	0.96 (0.051)
XOM	0.10 (0.014)	0.30 (0.018)	0.10 (0.016)	0.62 (0.017)	0.25 (0.000)	0.97 (0.026)	0.07 (0.019)	0.29 (0.031)	0.10 (0.029)	0.63 (0.029)	0.26 (0.029)	0.97 (0.045)
XRX	0.12 (0.014)	0.23 (0.009)	0.05 (0.007)	0.73 (0.007)	0.67 (0.000)	0.99 (0.012)	0.07 (0.029)	0.23 (0.027)	0.08 (0.023)	0.73 (0.023)	0.69 (0.025)	1.00 (0.037)

Estimated parameters and standard errors (in parenthesis) for the SPvMEM (left) and the univariate MEM (right).